Regimes and Long Memory in Realized Volatility *

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Abstract

We analyze the dynamics of realized variance and bipower variation of daily stock returns of 30 companies in the Dow Jones index using a threshold fractionally integrated autoregressive and moving average model. We find that the logarithmic transformation of the square root of realized variance and bipower variation processes can be characterized by high and low volatility regimes and that the persistence, long memory and variance differ in regimes. Also, we find that forecast models using realized variance perform better than models using bipower variation. The high volatility regime is explained better by realized variance than by bipower variation. We find that the two regime threshold fractionally integrated autoregressive and moving average model that accounts for different degrees of long memory, persistence and variance in two regimes outperforms the fractionally integrated autoregressive and moving average model commonly used in the literature. A new test based on posterior distributions of the mean squared forecast errors (MSE) is used for model selection.

1 Introduction

Volatility is an important measure in many financial applications such as option pricing, risk management and portfolio optimization. In addition volatility is a tradable asset where buyers and sellers gain or lose depending on volatility realized until maturity. Since the Chicago Board Options Exchange (CBOE) introduced option trading on the volatility of the S&P500 index (VIX), many volatility-based derivatives have been introduced in the option markets. Trading in volatility derivatives attracted many investors since it provided more tools for managing portfolio risk, option positions and exploiting the leverage effect of negative correlation between volatility movements and stock index returns.[1] As the volatility derivative markets grew, research on volatilities has expanded, and realized volatility based on high-frequency returns has become a popular measure of volatility.

Since Andersen and Bollerslev (1997) introduced the concept of realized volatility there has been considerable interest in its distribution, modeling, and forecasting. Fleming et al (2003) show the economic value of using realized volatility for portfolio optimization. A risk-averse investor can improve the performance of her mean-variance efficient portfolio when she switches from daily to intra-daily measure of variances and covariances for volatility timing. Andersen et al (2001a), and Andersen et al (2001b) provide distribution of logarithmic transform of realized volatility of foreign exchange returns and stock returns respectively. Andersen et al (2003) (ABDL 2003) used a fractional autoregressive model in order to account for the long memory properties of realized volatility and found that these forecasts are better than forecasts produced by many popular models of volatility, e.g. daily GARCH, RiskMetrics, and FIEGARCH. However spikes in realized volatility and nonlinearities in the dynamics are not incorporated in these models.

Barndorff-Nielsen and Shephard (2004) introduced bipower variation in order to separate jumps from the realized variance. Maheu and McCurdy (2002) emphasized nonlinear features of realized volatility for the DM/$ exchange rate and found that one day forecasts from a duration dependent Markov-switching ARMA model are better than from linear models. However their model is restrictive to two regimes with an unobservable state variable determining the regime and no analysis of long memory properties. Additional papers that used Markov-switching models for international equity returns and volatilities include Ang and Bekaert (2004, 2002), Ramchand and Susmel (1998), Guidolin and Timmermann (2002), and Liu and Maheu (2008) among others.

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[1]See e.g. Hafner and Wallmeier (2008) who analyzed variance swap positions for optimal portfolio investments.

As an approximation to a long memory ARFIMA process several papers used a heterogeneous autoregressive model (HAR) (see Corsi (2009), Liu and Maheu (2008) and McAleer and Medeiros (2008) among others). We do not consider this approximation for long memory, since we show an efficient way of joint estimation of the fractionally differenced parameters d and threshold parameters r in ARFIMA and TARFIMA models considered below. However, for the future research it would be interesting to compare HAR and ARFIMA models.

There are several papers that provide alternative measures of realized volatility adjusting for microstructure noise (see Bandi et.al (2008), Ait-Sahalia and Mancini (2008), etc). In the presence of microstructure noise if sampled at high frequency intervals the realized volatility can become a biased estimate of daily volatility since noise can be time dependent and correlated with closing prices. However, as suggested in Andersen et al. (2001) if the data is sampled at a lower frequency (they suggested 5 minutes) the realized volatility is a good estimate in practice. In this paper we do not study microstructure noise and possible adjustments, rather, we proceed with 5-minute intra-daily returns (excluding holidays and non-trading hours) obtained from the NYSE Trade and Quote (TAQ) database.

In this paper, we use a multiple regime threshold fractionally integrated autoregressive and moving average model (TARFIMA) model to analyze realized variance and bipower variation. The advantage of the TARFIMA model is that it is more flexible than a linear model since it is characterized by regimes with different persistence, long memory and variance, thus accounting for asymmetric dynamics in regimes. Compared to Markov-switching models threshold models use a threshold variable determining the regimes based on observable past level of volatility or some other observable variable[2] rather than random switching with an unobservable variable determining the state. Threshold models allow the explicit estimation of threshold levels of volatility that change with regimes.

The TARFIMA model can capture the difference in dynamics of high and low volatility regimes. One of the prominent features of the data are frequent spikes in realized volatility and bipower variation. Although bipower variation removes some jumps there are still many spikes in the data that can be observed in a regime of high volatility. The TARFIMA model introduced here is quite general because the parameters of ARMA, long memory and variance change with regime and because it allows us to find multiple threshold regimes.[3] In particular, we investigate nonlinearities in

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[2]One could use past level of the return rather than volatility as a threshold variable. This choice of the threshold variable may reflect the leverage effect, which is the asymmetric effect of negative and positive returns on volatility.

[3]Many recent papers use Threshold Autoregressive (TAR) models in economic and financial time series to model the dynamics of short-term interest rates, real exchange rates, unemployment rate,

the dynamics of realized daily variance and bipower variation of stock returns of 30 companies in the Dow Jones Index.

We find that the logarithms of two volatility processes can be characterized by two regimes of high and low volatility. The persistence, long memory and variance change with each regime. This model helps us understand better the dynamics of intra daily returns and improves forecasts compared to linear ARMA and fractional ARMA (ARFIMA) models since forecasts based on estimated thresholds regimes of high or low volatility can be predicted separately. This information indicates the difference in investors behavior in high and low volatility regimes. The high volatility regime is characterized by a higher variation in realized volatility which arises because of increased activity in the market. On the other hand the higher volatility regime may have higher transaction costs and thus higher long memory parameter. Also abrupt shifts in volatility may induce long memory characteristics. On the other hand the regimes of high volatility are short-lived and mean-reverting as we find from autoregressive roots.

We also show that short lived spikes in realized volatility captured by upper regimes are important in modeling and forecasting. We show that realized variance generally produces a better forecast than bipower variation for the stocks in the Dow Jones index, since the former contains more information about the spikes incorporated in the dynamics of a high volatility regime.

The TARFIMA model is estimated using Bayesian Markov Chain Monte Carlo (MCMC) algorithms with efficient jump as an extension of the algorithm in Goldman and Agbeyegbe (2007). The algorithm is extended for the fractional integration parameter d. The Metropolis-Hastings algorithm with efficient jump allows us to estimate jointly parameters of all regimes and threshold parameters. For model selection and specification several in-sample techniques are used such as information criteria, significance of parameters and estimation with and without truncation restrictions imposed on thresholds.

The out-of-sample forecast performance is used in this paper for comparison of TARFIMA, ARFIMA and ARMA models. We suggest comparison based on the distributions of mean squared errors (MSE). We illustrate a test of the dominance of the cumulative density for one model over other models. Using MCMC such a test is easy to implement and it incorporates the whole distribution of MSE rather than the point estimate frequently used in the literature for forecast evaluation.

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stock prices, production, and inventories. See e.g. Tsay (1989,1998); Hansen (1997); Koop and Potter (1999); Phann et al (1996); Forbes et al(1999); Goldman and Agbeyegbe (2007), Dufrenot et al. (2008) among others.

Our paper contributes to the literature of modelling realized volatility as the first to apply a TARFIMA model with estimated thresholds, changing long memory parameters and variance. We also provide a test for MSE to select the best model. We show that TARFIMA model provides a better fit to the realized volatility processes than the benchmark linear fractionally integrated model. So far in the realized volatility literature fractional autoregressive models are known to produce forecasts that are better than the forecasts of other models (ABDL 2003). Our results show that TARFIMA models tend to outperform ARFIMA models for realized volatility of individual equity returns in the DJ index. Thus TARFIMA models with several regimes may provide more benefit to investors in the areas of portfolio optimization and risk management.

The plan of the paper is as follows. In Section 2 we provide the TARFIMA and ARFIMA models and a Bayesian test of persistence, long memory and mean-reversion. In Section 3 we explain estimation results for logarithms of realized variance and bipower variation. In section 4 we perform forecasts and model comparisons for TARFIMA, ARFIMA and ARMA models. In Section 5 we give concluding remarks and plans for future research. Appendix with Bayesian posterior distributions and MCMC algorithms is in section 6.

2   Model

The term realized variance RV was introduced by Andersen and Bollerslev (1997) as the sum of squared intra-day returns:

$$RV_t = \sum_{\tau} r^2_{t+\tau\Delta}$$

where $r_{t+\tau\Delta}$ are returns for small intra-daily periods of length Δ within a day t. Five-minute returns are accepted by many studies as a norm.[4] Realized volatility is defined as the square root of realized variance $RV^{1/2}$.

Barndorff-Nielsen and Shephard (2004) suggested using the bipower variation BV in order to separate jumps from the total daily variation. With number of observations within a day going to infinity it is given by

$$BV_t = \pi/2 \sum_{\tau} |r_{t+(\tau-1)\Delta}||r_{t+\tau\Delta}|$$

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[4]Higher frequencies may result in microstructure problems (see Ait-Sahalia, Mykland and Zhang, 2005).

In the literature that measures separately the continuous part of total realized variance measured by BV_t and jumps $(RV_t - BV_t)$ the statistics for determining the significance of jumps is based on the size of the relative jump as defined by Huang and Tauchen (2005), $\left(\dfrac{RV_t - BV_t}{RV_t}\right)$. This implies that on the days when volatility is high the jump may not be detected if the size of the relative jump is not large, while on low volatility days a smaller increase in volatility may qualify as a jump.[5]

In our study we find that instead of separating BV_t and jumps we can look at several regimes of the realized variance and compare the dynamics of high and low volatility regimes. Since we also know from previous studies (e.g. ABDL 2003) that realized volatility exhibits long memory we propose using a threshold fractionally integrated model with regimes that are characterized by different time series dynamics, variance and long memory parameters in each regime.

The regimes are determined by an observed threshold variable. In our case it is the past level of realized variance, for example, the lagged realized variance above the threshold level will determine the high volatility regime[6]. In this model the highest volatility regime will be short-lived since overall variance is stationary. The current level of volatility will determine the forecast of the next period state (or regime) and will contain valuable information for portfolio diversification (see Ang and Bekaert (2002, 2004)).

In addition, the distribution of the logarithm of realized volatility may be better captured by a mixture of several normal distributions from several regimes rather than one normal distribution. The mixture of normal distributions can approximate well any distribution (Ferguson (1973)). Thus, the threshold model with several regimes is expected to better fit the unknown distribution of the data.[7]

In the next two subsections we describe the ARFIMA and TARFIMA models.

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[5]See Bollerslev et al. (2008) for the data example illustrating this point.

[6]Alternatively past level of returns could be chosen as a threshold variable.

[7]Although the logarithm of realized variance looks normal, statistical tests (e.g. Jarque Bera test) often reject normality.

2.1 Fractionally Integrated ARMA (ARFIMA) Model

Let $y_t = ln(RV_t^{1/2})$ or $ln(BV_t^{1/2})$. The fractionally integrated ARMA model (ARFIMA) is given by

$$(1 - B)^d y_t = x_t \gamma + u_t \tag{1}$$

$$u_t = \frac{\Theta(B)}{\Phi(B)} \epsilon_t$$

$$\Theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

$$\Phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

$$\epsilon_t \sim N\left(0, \sigma^2\right)$$

where d is a long memory fractional integration parameter, γ is a vector of regression parameters for x explanatory variables[8], σ^2 is variance (showing variation in realized volatility), $\Phi(B)$ and $\Theta(B)$ are p-th order and q-th order polynomials in the the backward shift (lag) operator B respectively. For $d > -1$ the difference operator $(I - B)^d$ has the binomial expansion

$$(I - B)^d = \sum_{j=0}^{\infty} \pi_j B^j \tag{2}$$

where

$$\pi_j = \prod_{0 < k \leq j} \frac{k - 1 - d}{k}, \quad j = 0, 1, 2, \cdots$$

When $p = q = 0$, Brockwell and Davis (1987) state that $\{y_t\}$ is a covariance stationary process if $-.5 < d < .5$. If $0 < d < 1$ the ARFIMA model is called a long memory process.

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[8]In the simplest case x is a constant term.

2.2 Fractionally Integrated Threshold ARMA Model (TARFIMA)

Let us consider the following dynamic threshold ARFIMA model (TARFIMA) with multiple regimes for logarithm of realized variance or bipower variation $y_t = ln(RV_t^{1/2})$ or $ln(BV_t^{1/2})$

$$(1 - B)^{d^{(j)}} y_t = x_t \gamma^{(j)} + u_t \tag{3}$$

$$u_t = \frac{\Theta^{(j)}(B)}{\Phi^{(j)}(B)} \epsilon_t \tag{4}$$

$$\epsilon_t \sim N\left(0, (\sigma^{(j)})^2\right)$$

where j=1,...,s+1 with s+1 regimes defined by

$$\begin{cases} j = 1 & y_{t-\delta} < r_1 \\ j = 2 & r_1 \leq y_{t-\delta} < r_2 \\ \vdots & \vdots \\ j = s+1 & y_{t-\delta} \geq r_s \end{cases} \tag{5}$$

We assume in the model above that there are s thresholds $r_1, r_2, ..., r_s$ separating $s+1$ regimes for y_t. The threshold variable y_{t-1} is an observable previous day volatility level with delay parameter, δ, set to one ($\delta = 1$).[9] The error term follows ARMA process $u_t \sim \text{ARMA}(p^{(j)}, q^{(j)})$, where $\Phi^{(j)}(B)$ and $\Theta^{(j)}(B)$ are $p^{(j)}$-th order and $q^{(j)}$-th order polynomials in the backward shift operator B respectively.

We note that each parameter in this model $\{\gamma^{(j)}, \phi^{(j)}, \theta^{(j)}, \sigma^{2\ (j)}\}$ takes $s+1$ values depending on the regime j where $y_{t-\delta}$ belongs. This allows a change in dynamics, persistence, and variance of realized volatility depending on regime.

2.3 Testing Nonstationarity and Mean Reversion

In the ARFIMA and TARFIMA models there are two sources of nonstationarity: fractional integration parameter d and the roots of $\Phi(B)$. Let ρ denote the maximum absolute value of the inverse roots of the AR parameters in the error term u_t. We

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[9]One can use any lag of volatility or a combination of several lags, but our analysis shows that the previous day provided the best fit for the model using MBIC criterion.

will test for the mean reversion of realized volatility y_t looking at the long memory parameter $d^{(j)}$ and at the autoregressive root parameter $\rho^{(j)}$ in each regime. A simple Bayesian unit root test using the maximum absolute value of inverted roots of AR parameters ρ was introduced in Goldman et al (2001) and we use the same method in this paper. It is important to test the null hypothesis of nonstationarity, since nonstationarity implies high persistence and long memory of time series. The nonstationarity hypothesis in our setting is jointly given by:

$$H_0 : \ \rho \geq 1 \quad \text{versus} \quad H_1 : \ 0 \leq \rho < 1$$

and

$$H_0 : |d| \geq 0.5 \quad \text{versus } H_1 : \ -0.5 < \ d < \ 0.5$$

Although covariance stationarity requires $d < 0.5$ there is still mean reversion if values of $0 < d < 1$ and the AR process is stationary. Diebold and Inoue (2001) define a long memory mean-reverting fractionally integrated process for $0 < d < 1$. Mean reversion for $0.5 < d < 1$ was illustrated for real exchange rates in Diebold et al (1991) and formally proved for ARFIMA(p, d, q) time series in Chung (2001). Both papers showed that impulse response functions converge to zero.

In the Bayesian approach it is common to use the highest posterior density intervals (HPDI) for hypothesis testing. One could construct a joint HPDI involving both parameters (e.g. for $\rho + d^2$), however, intuitively it is easier to interpret these two parameters separately. If the 95% HPDI includes 1 for ρ or includes 1 for d we would not reject a unit root. If the HPDI for ρ does not include 1 and HPDI for d is within [-0.5,0.5] we would conclude that the process is stationary. If the HPDI for ρ does not include 1 and HPDI for d does not include 1 we would conclude that the process is mean-reverting even if $d > 0.5$. Finally if the HPDI for d is positive and does not include 0 we conclude that the time series exhibits long memory.

2.4   Model Choice

After TARFIMA, ARFIMA and ARMA models are estimated using Markov Chain Monte Carlo (MCMC) algorithms with efficient jump, explained in Appendix, we compare models based on the distribution of their out-of-sample mean squared errors (MSE). MSE is the most common overall measure of forecast accuracy measuring average squared deviations of forecasted values from observed values.[10] Using MCMC

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[10]The mean squared errors are commonly used in the classical statistics. The Bayesian analogue of this popular measure is used in this paper.

we can get the posterior distributions of the MSE for each model as explained below. The model with MSE distribution that dominates all other distributions using cumulative density function (CDF) is chosen over other models.[11]

The distributions of mean squared errors are obtained as follows. We estimate the model for the sample of data 1...T and make h-step ahead repeated forecasts based on the fixed sample estimates of parameters. At each $i-$th draw of all parameters $\theta^{(i)}$ we find the predicted values of

$\hat{y}_{T+1}^{(i)}|(y_1, ..., y_T, \theta^{(i)})$
$\hat{y}_{T+2}^{(i)}|(y_1, ..., y_T, \hat{y}_{T+1}^{(i)}, \theta^{(i)})$
$\vdots$
$\hat{y}_{T+h}^{(i)}|(y_1, ..., y_T, \hat{y}_{T+1}^{(i)}, ..., \hat{y}_{T+h-1}^{(i)}, \theta^{(i)})$.

Using MCMC algorithms we obtain the draws of MSE for the predicted values of $\hat{y}_{T+j}, \ \ j = 1, \cdots, h$:

$$MSE^{(i)} = \frac{1}{h} \sum_{j=1}^{h} (\hat{y}_{T+j}^{(i)} - y_{T+j})^2$$

where $\hat{y}_{T+j}^{(i)}$ is the i-th MCMC draw of the predicted value at time $T + j$ and y_{T+j} is the realized value at $T + j$.

The mean forecast errors (ME) are average differences between predicted values from MCMC draws and actual values of $y_{T+j}, j = 1, ..., h$:

$$ME^{(i)} = \frac{1}{h} \sum_{j=1}^{h} (\hat{y}_{T+j}^{(i)} - y_{T+j}).$$

The mean error measures average bias (positive or negative) and is another measure of accuracy. Using MCMC we obtained the distribution of MSE and ME for each out-of-sample observation.

The CDF of the distribution of MSE closest to zero indicates which model has the best forecast accuracy. The model selection is based on min MSE for the posterior modes or medians of their distributions. Since the distribution of MSE taking only positive values is heavily skewed to the right it makes sense to use the mode or median rather than the mean of distribution.[12]

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[11]We note here the advantage of using a Bayesian approach where posterior distributions of any parameters of interest are easily obtained using MCMC draws as explained below. It is common in current forecasting literature to provide the distribution of forecasts rather that a point forecast.

[12]The mean is highly affected by skewness compared to median or mode of asymmetric distributions.

If we have m posterior predictive densities for repeated 1 day or 5 day forecasts we can find the average of the modes (or medians) of all distributions. For example:

$$\text{MMSE}_i = \frac{1}{m} \sum_{j=1}^{m} \text{mode}_j(\text{MSE}|M_i)$$

where MMSE_i is the average of modes of forecasts for model M_i. We choose the minimum of MMSE:

$$\min_{i \in \{1,..,3\}} \text{MMSE}_i$$

among 3 competing models.

3 Data Analysis

The models are estimated for log realized volatilities and bipower variation of stock returns for 30 companies in the Dow Jones index. The data obtained from the TAQ database are 5-minute returns from January 4, 1993 to December 31, 2004. The corresponding daily realized variance and bipower variation have 3021 observations of trading days except for five companies for which the data starts later.[13] We used data from 1/1/1994 to 7/29/2003 for estimation and the last 360 observations (7/30/2003-12/31/2004) were left for out-of-sample forecast evaluation.

Figure 1 shows the realized volatility $RV^{1/2}$ and the square root of bipower variation $BV^{1/2}$ for three stocks: GE, MSFT and HPQ. These three graphs represent typical features of volatility. We can observe multiple spikes in realized volatility and bipower variation with a slight difference in the magnitude. Bipower variation is slightly smoother but still captures most of the spikes as can be seen from the graphs. The difference between the graphs for the three stocks is the magnitude of spikes. For either $RV^{1/2}$ or $BV^{1/2}$ we observe the most extreme spike for MSFT on April 1, 1997 and frequent smaller size spikes on other dates. For GE we observe two extreme spikes of lower magnitude than for MSFT with many smaller spikes and for HPQ there are no extreme spikes; we observe only frequent spikes of smaller size. The advantage of using the TARFIMA model with two regimes is that it explicitly allows us to account for spikes, that can be defined once volatility is above a certain threshold level.

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[13]The five companies with less than 3021 trading days are HPQ (671 obs), MO (3023 obs), C (3010 obs), XOM (1278 obs), and VZ(1130 obs) as can be seen from Tables 1 and 2.

Tables 1 and 2 give summary statistics of the $ln(RV^{1/2})$ and $ln(BV^{1/2})$ for 30 companies showing number of observations (number of trading days), mean, standard deviation, first order autocorrelation, skewness, kurtosis, and estimated fractional-differencing parameter d^e with asymptotic standard error using Geweke and Porter-Hudak (GPH) (1983) log - periodogram. Following ABDL 2003 using d^e we can examine long memory properties of realized volatility and bipower variation. In the following sections we compare estimated results and forecasting performance of ARFIMA and TARFIMA models. We see that the GPH fractional parameter for 30 companies ranges from 0.373 to 0.553 with standard errors 0.027 and 0.042 respectively. For the $ln(BV^{1/2})$ the results are almost identical to $ln(RV^{1/2})$. The periodogram estimates of d^e all seem to indicate long memory processes with statistically significant $d > 0$. However, we also notice that most series are highly autocorrelated with the first degree autocorrelation AR(1) around 0.6-0.7. The GPH estimator is simple to apply, but as pointed out in Baillie (1996) among others it is substantially biased in the presence of significant autocorrelation. Therefore, we prefer to use joint estimation of the fractional parameter with ARMA parameters using MCMC as described in the previous section.

The results of the estimated TARFIMA model (3)-(5) for 30 equity returns are given in Table 3. The second and third columns give estimated thresholds with posterior standard deviations in brackets for $RV^{1/2}$ and $BV^{1/2}$ correspondingly. The forth and fifth columns give the percent of observations in the regime of low volatility below the threshold. The sixth to ninth columns give the estimated maximum absolute value of autoregressive roots in each regime, where regime 1 is below the threshold and regime 2 is above the threshold. The last four columns give estimates of fractional integration parameters d for each regime of $RV^{1/2}$ and $BV^{1/2}$. We provide mean and standard deviation for each parameter in each regime.[14] Using MBIC information criterion we find the number of regimes and orders of ARMA model in each regime. For all 30 companies we found that models with 2 regimes are preferred to three regimes and the orders of ARMA(1,1) were selected for the TARFIMA model in all regimes.[15]

The dynamics in the lower regime are generally more persistent than in the upper regime as can be seen from the maximum absolute value of the AR roots given in Table 3. For regime 1 the AR root ranges from 0.158 to 0.982 and the AR coefficient ϕ is positive and significant (using 95% HPDI) for 17 out of 30 stocks. For the rest of the stocks the AR parameter in regime 1 is positive and insignificant except for 1

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[14]The 95% HPDIs are omitted here to save space. They are available from the authors on request.

[15]The results of order (1,1) are not surprising since the ARFIMA(p,d,q) model is equivalent to the infinite lag ARMA model. Thus, the fractional integration parameter d takes care of higher order lags.

stock when it is negative.[16] For regime 2 the AR root presented in Table 3 ranges from 0.103 to 0.527 with the AR parameter always insignificantly different from zero and negative (in one case). Thus regime 2 shows the mean reversion in the autoregressive part once volatility is above the threshold level. The unit root hypothesis using 95% HPDI (as explained at the end of the section 2) is rejected for all series as all 95% HPDIs are below one for the AR parameter. The implication of this result is that in most cases abrupt spikes in volatility are not persistent and volatility quickly reverts back to the lower regime of normal volatility. By normal volatility we mean volatility below the estimated threshold values given in the second column of Table 3. The average value of the threshold for volatility is 0.859 and the corresponding average percentage of observations in the lower volatility regime is 80%. This confirms that most of the time volatility stayed at a normal level and spikes form a significant regime accounting for 20% of observations.

It is interesting that results for the bipower variation are very similar. The level of the threshold is slightly lower due to the removal of a few spikes by the definition of BV. The average value of the threshold is 0.754 which corresponds to 78% of observations in the lower regime. For the lower regime the AR parameter is positive and significant in 19 cases and insignificant for the rest of the stocks. For the upper regime the AR parameter is insignificant in 28 cases, significant and negative in one case and positive in one case. For all regimes we reject the unit root for AR parameters.

The long memory fractional integration parameters are positive and significant for all 30 stocks in both regimes showing that both $ln(RV^{1/2})$ and $ln(BV^{1/2})$ exhibit long memory in both regimes. If we compare two regimes the long memory parameter d is on average higher in regime 2 than in regime 1. The 95% HPDI for parameter d is within the interval $(0,1)$ for all cases except one.[17] The 95% HPDI is within the interval $(0, 0.5)$ only for 4 stocks in regime 2. Therefore, the long memory parameter shows high persistence although as discussed in section 2.3 the volatility is still mean-reverting. Long memory results for the bipower variation are very similar to the realized volatility results.

We also found that σ^2, the variance of log volatility $ln(RV^{1/2})$ in equations (3)-(5), is in most cases significantly higher for the upper regime compared to the lower

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[16]Results for AR parameters ϕ are omitted to save space–they are similar to $rho =$ the maximum absolute value of AR root (presented in Tables 3,6, 9) except for the sign of ϕ when negative. Results are available from the authors on request.

[17]For XOM the 95% HPDI in regime 2 is [0.53, 1.09] with the upper bound very close to 1. Since number of in-sample observations for XOM is about one third of observations for most stocks and the estimation of TARFIMA model includes many parameters it results in higher standard deviation and wider confidence interval.

regime. Spikes in realized volatility are characterized by a higher variance of volatility, while a lower regime of volatility has more persistence and smaller variance.

The results of estimated ARFIMA model (1) are given in Table 6. The second and third columns give estimated maximum absolute value of autoregressive roots for $RV^{1/2}$ and $BV^{1/2}$ correspondingly. The last two columns give estimates of fractional integration parameters d. For all 30 companies the orders of ARMA(1,1) were selected using MBIC criterion. Orders are the same as for the TARFIMA model.

Compared to the TARFIMA model the results for a one regime ARFIMA model are different in several respects. First, the AR root is higher for the ARFIMA model, showing higher persistence compared to both regimes of the TARFIMA model. The upper bound of the maximum autoregressive root is very close to 1 for 21 stocks out of 30 stocks.[18] Thus it is very close to unit root rejecting it only marginally. On the other hand in all 21 cases the 95% HPDI is $d < 0.5$ and for the remaining 9 cases $0.5 \leq d < 1$. Long memory parameter is statistically insignificant ($d = 0$) for 3 firms. Overall, the results show that there is more persistence in AR parameters and less degree of a long memory for the ARFIMA model compared to a two-regime TARFIMA model. For the bipower variation the maximum AR root is very close to 1 for 16 stocks and is positive and significant for the rest. On the other hand the long memory parameter is less than 0.5 for 11 cases and is zero for 1 case. Thus both realized variance and bipower variation show higher persistence in AR parameter and less degree of long memory in ARFIMA model compared to two-regime TARFIMA model.

Finally in the third column of Table 9 we shows the results for the maximum autoregressive root for a simple benchmark ARMA(1,1) model for $ln(RV^{1/2})$.[19] The results for $ln(BV^{1/2})$ are omitted as they are almost identical. The average AR parameter is 0.959 and the upper bound of the 95% HPDI is between .9460 and .9999 for 30 stocks. Although the unit root is rejected the persistence in AR is higher for the simple ARMA(1,1) that ignores long memory compared to ARFIMA or TARFIMA models.

This information indicates the difference in investors behavior in high and low volatility regimes. The high volatility regime is characterized by higher variation in realized volatility which arises because of increased activity of the market. On the other hand the higher volatility regime may have higher transaction costs and thus a higher long memory parameter as we find for the TARFIMA model in Table 3. As observed in the literature abrupt shifts in volatility may induce long memory

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[18]The 95% HPDI upper bound is above 0.99 in 16 cases and above 0.98 in 21 cases.

[19]The orders of ARMA model were set to (1,1) for simplicity. Higher orders were significant up to a month as realized volatility series exhibit long memory.

characteristics (see Andersen and Bollerslev (1998)). Dufrenot et al. (2008) among others discuss how herding mechanism may result in changing volatility and long memory. On the other hand the regimes of high volatility are short-lived and mean-reverting as we see from AR roots.

In the following section we discuss the forecasting performance of each model.

4   Forecasting and Model Comparison

In this section we compare the forecasting performance of the TARFIMA model with linear ARFIMA and ARMA models using realized volatility and bipower variation. Estimates of each model were first obtained for the in-sample period 1/1/1994 to 7/29/2003 (given in Tables 3, 6 and 9) and then used for repeated 1 and 5 day out-of-sample forecasts for the last 360 days (7/30/2003-12/31/2004). ABDL 2003 used the fractionally integrated autoregressive model in order to account for long memory properties of realized volatility and found that their forecasts are better than forecasts produced by many popular models of volatility, e.g. daily GARCH, RiskMetrics, and FIEGARCH. Our TARFIMA model allows use of the information about the thresholds to predict regimes of high or low volatility and different dynamics in each regime.

In Tables 4, 5, 7, 8 and 9 we present average posterior modes and medians of mean squared errors (MSE) and mean errors (ME) for 1 and 5 day forecasts for all models. Tables 4 and 5 show results for $RV^{1/2}$ and $BV^{1/2}$ for the TARFIMA model. Tables 7 and 8 show results for the ARFIMA model and Table 9 shows results for ARMA model[20] for $RV^{1/2}$.

If we compare the results for $RV^{1/2}$ and $BV^{1/2}$ models we make the following observations. One day average modes of MSE for the TARFIMA $RV^{1/2}$ model are for all companies smaller or equal to the 1 day average MSE modes of the TARFIMA $BV^{1/2}$ model showing that $RV^{1/2}$ produces a more accurate forecast. For the MSE at median again the $RV^{1/2}$ forecasts outperform $BV^{1/2}$ except for 2 cases. Similarly for the ARFIMA model in most cases MSE is smaller for realized volatility models compared to bipower variation models. MSE is generally smaller for models with $RV^{1/2}$ since realized volatility contains more information about spikes. At the same time the ME shows a negative bias for $BV^{1/2}$ as can be seen from 1 day ME in Table 4 for TARFIMA models. Since $BV^{1/2}$ excludes some spikes it produces a more smooth forecast with smaller values of volatility compared to forecasts from $RV^{1/2}$. Similar

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[20]As we mentioned earlier the difference in results between $RV^{1/2}$ and $BV^{1/2}$ for the ARMA model is negligible.

negative bias for $BV^{1/2}$ can be seen for the ARFIMA model in Table 7 where most ME are negative for bipower variation. Overall realized volatility generally produces better forecasts than bipower variation judged by MSE and ME. Therefore we use realized volatility for further forecast comparisons between models.

Table 10 shows the relative mean squared errors for 1-step and 5-step prediction. Relative MSE is normalized relative to the linear ARMA model. Using the average modes and medians of MSE listed in Table 10 we can compare the forecasting performance of TARFIMA, ARFIMA and ARMA at 1 and 5 day forecasting horizons. Table 11 summarizes the comparison for the three models and reports the number of cases when a particular model is selected based on minimum average mode (or median) of MSE as defined in section 2.4. For the 1-day forecasts using modes of MSE we see that the TARFIMA model outperforms the benchmark ARMA model in all cases and outperforms the ARFIMA model in 28 cases. Only in 2 cases does the long memory ARFIMA model have a lower average mode of mean squared errors. Using medians of MSE, the TARFIMA model outperforms ARFIMA in 19 cases and ARMA in 29 cases. The ARFIMA model outperforms TARFIMA in 10 cases. For the 5-day forecasts using either modes or medians of MSE we see that the TARFIMA model outperforms ARFIMA in 21 cases and ARMA in all cases. The ARMA(1,1) model (M3) is always dominated by either ARFIMA or TARFIMA models except 1 case for 1 day average median MSE. Since the distribution of MSE is highly skewed to the right (MSE takes only positive values) it it better to use modes of distributions rather than means or medians which are biased to the right from the peak of the distribution. However, we provide medians for robustness and still see that the TARFIMA model dominates other models in most cases. The lower part of Table 11 shows models selected for three sample companies: TARFIMA is selected for GE and MSFT and ARFIMA is selected for HPQ.

Better out-of-sample forecasts do not always imply better in-sample fit to which we turn next. The MBIC criterion in most cases favors the simplest ARMA(1,1) model which has the smallest number of parameters. However, by increasing the number of lags which happen to be significant due to the presence of long memory, MBIC is not favoring linear ARMA(p, q) models with a large number of lags. For the ARFIMA model, which can be represented as an infinite order ARMA model, MBIC is always higher than for the simple ARMA(1,1) model. However, for the TARFIMA model, MBIC is improved in 6 cases compared to the simple ARMA(1,1) and in most cases compared to the ARFIMA model even though the TARFIMA model has more than twice parameters compared to ARFIMA. Thus the TARFIMA model shows better in-sample fit than the ARFIMA model even with high penalty for extra parameters. We would not select the simple ARMA(1,1) model, since it is misspecified in terms of number of lags and inferior out-of-sample performance.

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The TARFIMA model forecasts are generally superior to linear models because of information about the threshold and regime conditioned on which the forecast is made. As discussed in section 3 the difference in dynamics reflects the difference in investors' behavior in high and low volatility regimes.

Tables 10 and 11 are summary statistics of forecast performances. They do not give the whole pictures of the forecasts. Accordingly, we present the graphs of the forecasts as compared to actual volatilities for three selected cases: GE, MSFT, and HPQ.

Figure 2 shows out-of-sample posterior means of 1 and 5 day forecasts for $RV^{1/2}$ of GE, MSFT and HPQ. The in-sample data was presented in Figure 1. We can notice that 1 day ARFIMA forecasts of volatility for GE lie above TARFIMA and ARMA forecasts. ARFIMA forecasts have a positive bias relative to the actual data on the graph. We can also notice a negative bias for the ARMA model for 1 day HPQ forecasts and a positive bias for the ARMA model for 5 day GE and 5 day MSFT forecasts. For other graphs the forecasts look very similar for the three models.

Figure 3 shows posterior probability densities (PDFs) for standardized logarithms of realized volatility forecasts for three selected stocks (GE, MSFT and HPQ).The distributions are close to normal with slight asymmetries. The forecast residuals can be approximated as normally distributed. Modelling the error term as normally distributed is a fairly standard assumption for the $log(RV^{1/2})$ in the literature (see Andersen et al. (2001a, 2001b, 2003) and Thomakos and Wang (2003) among others). Although the Jarque Bera (JB) test rejects normality for the logarithm of realized volatility data it was shown in Thomakos and Wang (2003) that the JB test is oversized when the series exhibits long memory or autocorrelation and results in more bias than other tests of normality. Since we found long memory and autocorrelation of realized volatility for all series the JB test is not appropriate. Other tests may or may not reject normality and could also be sensitive to autocorrelation. For example, we performed the Kolmogorov-Smirnov test and concluded that normality is not rejected in 16 cases out of 30, compared to the JB test where normality is rejected in each case. The residuals of TARFIMA and ARFIMA models (correcting for autocorrelation) are closer to the normal distribution than original data. We note the advantage of using the TARFIMA model which provides unconditional distribution of the error term as a mixture of normal distributions with two different variances corresponding to upper and lower regimes. Thus theoretically it should give a better approximation to the true distribution of the data.

In Figure 4 we show posterior cumulative densities (CDFs) of the medians of MSE for the ARFIMA and TARFIMA models.[21] The CDFs are calculated on the same set

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[21]Using modes of MSE instead of medians provides similar results.

of horizontal points for MSE using the first degree Taylor approximation.[22] From the visual comparison of CDFs we see that the distribution of medians of MSE is closer to zero for the TARFIMA model for the 5 day forecasts of GE and for the ARFIMA model for the 5 day forecasts of HPQ. In other cases TARFIMA and ARFIMA CDFs look very close. Figure 5 shows the PDFs of the differences between the CDFs in Figure 4 for ARFIMA and TARFIMA models. The pdfs of the difference provide a statistical test of the difference in the distributions in Figure 4. Positive values indicate that the ARFIMA model outperforms TARFIMA in MSE, while negative difference shows the superiority of the TARFIMA model. If x is the distance between CDF of ARFIMA and TARFIMA, then we can define the probability of the TARFIMA model being selected over ARFIMA as given by P(x<0). The results for three companies are as follows. The probability that the TARFIMA model is superior is 0.53 and 0.99 for GE for 1 and 5 day forecasts respectively. Thus, TARFIMA is a preferred model for GE especially for a longer-term forecast where the distribution of MSE of TARFIMA dominates that of ARFIMA with 99% probability. For MSFT the probability of selecting TARFIMA is 0.51 and 0.52 for 1 and 5 day forecasts. We select TARFIMA models but probabilities of ARFIMA and TARFIMA models are close to 0.5 thus giving similar MSE distributions. For HPQ the probabilities of TARFIMA are 0.17 and 0.01, thus we select the ARFIMA model with high probability in each case. These results confirm the results in Table 11 where TARFIMA was selected for GE and MSFT and ARFIMA for HPQ based on average of MSE medians or modes.

We note that the ARFIMA model works better than TARFIMA for smaller samples of observations (as the case of HPQ with 671 observations and only 311 in-sample observations used for estimation). Considering that the TARFIMA model has 2 regimes the estimates of parameters are less precise for each regime where the sample is small (for HPQ the upper regime has less than 20% or 60 observations) compared to cases when the data has more observations.[23] The TARFIMA model has more than twice the number of parameters than the ARFIMA model, thus it is better to use large samples for the TARFIMA model. On the other hand the TARFIMA model captures the differences in persistence, long memory and volatility in two regimes, thus it can add information for better forecasts in large samples, especially for multi-

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[22]We thank Evgeny Goldman for suggesting the following way to provide a common grid for two CDF graphs. Let a_i and b_i denote samples of x's for two CDF graphs. The grid of common x's is given by $x_i = \Delta x i$, $i = 1, ... nt$, where number of points in the grid $nt = min((a_{max} - a_{min})/\Delta, (b_{max} - b_{min})/\Delta))$. The first degree Taylor approximation for two CDF's is given by $F_a(x_i) = F(a_i) + (x_i - a_i)\frac{F(a_i)-F(a_{i-1})}{a_i-a_{i-1}}$ and $F_b(x_i) = F(b_i) + (x_i - b_i)\frac{F(b_i)-F(b_{i-1})}{b_i-b_{i-1}}$ correspondingly.

[23]Number of observations in each regime determines the precision of estimated parameters. We restrict number of observations in each regime to be at least 10% of the whole sample.

day forecasts compared to 1 day forecasts if sample size is sufficiently large to allow better precision in estimation of parameters.

The method of model selection using the posterior distribution of the differences between CDFs of MSE is new in this paper. It is better than methods based on point MSE estimate comparison since the latter does not account for uncertainty of MSE estimation. The posterior predictive densities of forecasts and posterior distributions of MSE provide information about uncertainty and tails of forecasts and provide straightforward tests of model selection based on the dominance of the distribution of MSE rather than point estimates of MSE. This is an advantage of the Bayesian MCMC approach used in this paper since posterior distributions of parameters and functions of interest are easily obtainable from MCMC chain.[24]

The model choice based on MSE distributions was used in this paper based on the popularity of MSE for forecasts comparisons. The CDF comparison test explained above can be applied in a straightforward manner to alternative measures of forecast accuracy, such as mean error or mean absolute error. Moreover we can compare more than two models based on their CDF distributions and rank them using probabilities of selecting each model as above for TARFIMA and ARFIMA. In our case the ARMA(1,1) model was always inferior to other models so we did not present its CDF.

In the classical approach it has become common to use the Diebold and Mariano (1995) test of superior predictive ability with many simulated forecasts for obtaining the distribution of MSE. In our future research we would like to compare the Bayesian approach of model selection with the Diebold and Mariano (1995) test.

5 Conclusion

In this paper, we used a threshold fractionally integrated autoregressive and moving average (TARFIMA) model to analyze the dynamics and forecasts of realized variance and bipower variation. We find that the logarithmic transform of volatility processes of 30 stocks in the Dow Jones Index can be characterized by regimes of high and low volatility. The persistence, long memory and variance of volatilities change with each regime.

Also, we find that models using realized variance perform better in forecasts than models with bipower variation. Realized variance allows more information about the spikes and incorporates that information in the dynamics of high volatility regime.

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[24]The GAUSS codes are available from the authors.

In addition, we also compared the forecast effectiveness of the TARFIMA model with ARFIMA and ARMA models. We find that the TARFIMA model that accounts for a different degree of long memory, persistence and variance in two regimes in most cases performs better than the ARFIMA model and the ARMA model. A new test based on the posterior distributions of mean squared forecast errors (MSE) is used for model comparison.

Given that the ARFIMA model is known to produce forecasts that are better than the forecasts by the ARMA and other models (ABDL 2003), our results show that TARFIMA models may provide more benefit to investors in the areas of portfolio optimization and risk.

Future research may be directed to studying how TARFIMA models can better accommodate jumps, for example, using market news and individual company news announcements. We could apply Lee and Mykland (2008) nonparametric tests for jumps since they showed that their test outperforms tests based on BV. TARFIMA models can be applied to other financial markets and used for identifying regimes of high volatility. Also we can can compare our Bayesian approach of model selection with Diebold and Mariano (1995) and other tests in the classical approach.

The final note is that for the periods of extreme volatility, such as during a financial crisis, the TARFIMA model with regimes based on past volatility intuitively seems to be more appropriate than linear models as TARFIMA explicitly accounts for the dynamic properties of a high volatility regime. The use of the Markov-switching model with long memory seems to be not as appropriate since its regimes are randomly determined by an unobservable variable, while for the TARFIMA model regimes are determined by the previous day level of volatility or some other observable variables. As more intra-daily data covering the financial crisis becomes available it will be interesting to compare the TARFIMA model with linear models.

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6 Appendix

Bayesian Estimation of ARMA, ARFIMA and TARFIMA Models

Let the prior probability density function for the TARFIMA model be given by

$$
\begin{aligned}
\pi(\gamma, \phi, \theta, \sigma^2, r, d) \quad \propto \quad & \prod_{j=1}^{s+1} \mathrm{N}(\gamma_0^{(j)}, \Sigma_\gamma^{(j)}) \times \mathrm{N}(\phi_0^{(j)}, \Sigma_\phi^{(j)}) \times \mathrm{N}(\theta_0^{(j)}, \Sigma_\theta^{(j)}) \times IG(\nu_0^{(j)}, \delta_0^{(j)}) \\
\times \quad & I\left(r^{(j)} \in [r_{low}^{(j)}, r_{up}^{(j)}]\right) \times I\left(d^{(j)} \in [-0.5, 1]\right)
\end{aligned}
\tag{6}
$$

where $\gamma, \phi, \theta, \sigma^2$ and d are

$$
\begin{aligned}
\gamma &= (\gamma^{(1)}, \cdots, \gamma^{(s+1)}) \\
\phi &= (\phi_1^{(1)}, \cdots, \phi_{p^{(1)}}^{(1)}, \cdots, \phi_1^{(s+1)}, \cdots, \phi_{p^{(s+1)}}^{(s+1)}) \\
\theta &= (\theta_1^{(1)}, \cdots, \theta_{q^{(1)}}^{(1)}, \cdots, \theta_1^{(s+1)}, \cdots, \theta_{q^{(s+1)}}^{(s+1)}) \\
\sigma^2 &= ((\sigma^{(1)})^2), \cdots, (\sigma^{(s+1)})^2) \\
d &= (d^{(1)}, \cdots, d^{(s+1)})
\end{aligned}
$$

We assume a uniform prior for threshold parameters r where each $r^{(j)}$ is constrained so that minimum $m\%$ of observations are within each regime. We use $m = 10\%$ of total number of observations as a minimum sample size in each regime.[25] We also assume uniform prior distribution for d in the parameter space between [-0.5, 1]. Other parameters have proper normal-inverted gamma priors with large variances in normal distributions.[26]

The posterior pdf is

$$
p(\gamma, \phi, \theta, \sigma^2, r, d | Y, X) \propto \pi(\gamma, \phi, \theta, \sigma^2, r, d) \prod_{j=1}^{s+1} \prod_{t \in T_j} \frac{1}{\sigma^{(j)}} \phi\left(\frac{(1-B)^{d^{(j)}} y_t - g(Z_t))}{\sigma^{(j)}}\right)
\tag{7}
$$

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[25]m depends on the number of observations, if the sample size is small higher m is recommended. For example, Koop and Potter (1999) used 15% as a minimum sample size in each regime.

[26]This priors are standard in the literature for ARMA models (see Chib and Greenberg (1994) among others).

where for every $t \in T_j = \{t : r_{j-1} \leq z_{t-\delta} < r_j\}$

$$
\begin{aligned}
u_t &= (1-B)^{d^{(j)}} y_t - x_t \gamma^{(j)} \\
\epsilon_t &= (1-B)^{d^{(j)}} y_t - g(Z_t) \\
g(Z_t) &= x_t \gamma^{(j)} - \sum_{i=1}^{p^{(j)}} \phi_i^{(j)} u_{t-i} - \sum_{i=1}^{q^{(j)}} \theta_i^{(j)} \epsilon_{t-i}
\end{aligned}
$$

MCMC algorithms for ARMA model were developed by Chib and Greenberg (1994). The autoregressive AR and variance parameters are drawn using a Gibbs sampler and moving average MA parameters are drawn using a Metropolis-Hastings algorithm. Chib and Greenberg (1994) (as well as Nakatsuma (2000)) use the constrained nonlinear maximization algorithm in the MA block. Alternatively one can use a Metropolis-Hastings algorithm with a random walk Markov Chain as was done e.g. in Goldman and Tsurumi (2005). The random walk draws speed up the computational time of the MCMC algorithms without losing much of the acceptance rate of of the Metropolis-Hastings algorithm. A simple intuitive explanation of the Metropolis-Hastings algorithm is given in Chib and Greenberg (1995).

In the literature for ARFIMA models there are several ways to handle the fractional difference $(1-B)^d y_t$. Sowell (1992) used the exact maximum likelihood. Hauser (1999) compares Sowell's procedure with others and shows Sowell's procedure tends to dominate the others. Bayesian estimation of ARFIMA models was done by Koop *et. al.* (1997), Gil-Alana (2001), and Gouskova (2002), among others. Shimizu and Tsurumi (2009) compare several procedures from a Bayesian perspective. They show that when the sample size is large the posterior pdf of d is insensitive to the choice of n_{lags} where n_{lags} is the truncation point of the infinite series (2).

In this paper the ARFIMA and TARFIMA models are estimated using Metropolis-Hastings MCMC algorithms with efficient jump. In order to estimate threshold parameters $r = (r_1, ..., r_s)$ and fractional difference parameters $d = (d_1, ..., d_{s+1})$ we employ an algorithm with the efficient Metropolis jumping rule that is decribed in Gelman et al. (2004) and was used for a threshold ARMA model in Goldman and Agbeyegbe (2008). The algorithm described below allows efficient simultaneous estimation of multiple thresholds and fractional difference parameters.[27]

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[27]Compared to existing methods of estimating multiple thresholds this method is more efficient, since we avoid estimation on a grid of points. Simultaneous estimation of multiple thresholds and difference parameters using many dimensions of grid search is virtually impossible. The Bayesian analogue of a grid search is a Griddy Gibbs sampler withing MCMC which was done for a single threshold parameter in Phann et al. (1996).

We briefly describe the algorithm for the case of two and three regimes. We estimate parameters in blocks: (i) regression parameters, γ; (ii) AR coefficients ϕ, (iii) MA coefficients θ; (iv) variance σ^2 parameters; (v) threshold parameters r, (vi) fractional difference parameters d.

We describe the procedure below.

(i) Choose initial values for γ, ϕ, θ, σ^2, r, d. Start from crude estimates of mean or mode of the posterior distribution. Generally it is hard to find a good approximation for mean or mode of threshold distribution because of an unconventional shape of the likelihood function. For starting values we simply divide the sample into regimes with equal samples. In case we have two regimes we use the mean value of y_t as a starting point $r^{(0)}$, if there are three regimes we divide sample into three equal subsamples to find starting values $r_1^{(0)}$ and $r_2^{(0)}$. Using initial values for d in each regime the series y_t is fractionally differenced using the binomial expansion (2) with some limited number of lags n_{lags}.

Let the started points be denoted by $\gamma^{(0)}$, $\phi^{(0)}$, $\theta^{(0)}$, $\sigma^{2\,(0)}$, $r^{(0)}$, $d^{(0)}$. Given initial values $r_1^{(0)}$, $r_2^{(0)}$ the samples $\{y_t\}$ and $\{x_t\}$ are separated into regimes based on $y_{t-\delta}$:

$$\begin{cases} j = 1 & y_{t-\delta} < r_1 \\ j = 2 & r_1 \leq y_{t-\delta} < r_2 \\ j = 3 & y_{t-\delta} \geq r_2 \end{cases}$$

(ii) Once the data are separated into regimes given thresholds r the model is transformed into an arranged ARFIMA model[28]. After y_t is differenced using fractional parameters d in each regime and using equation (2) with fixed n_{lags} the model becomes an arranged ARMA model (with different parameters in each regime), the estimation of which using MCMC is standard (see Chib and Greenberg (1994) who used independence chains or Goldman and Tsurumi (2005) who used random walk algorithm). We draw parameters of ARMA in each regime block by block using random walk Markov Chain. We draw parameters of variance σ^2 using the standard inverted gamma distribution. The acceptance rates of ARMA blocks are controlled by multiplying the variance of the proposal density with a scaling constant.

(iii) **Threshold parameters**
We use the following procedure with efficient jump.

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[28]We construct an arranged ARFIMA model in a similar way as Tsay (1989) and others constructed arranged autoregressive model sorting data y_t by regimes.

Let the superscript, (i), denote the i-th draw. Each threshold parameter $r_j^{(i)}$ $(1 \leq j \leq s)$ can be drawn either in a separate block given other threshold parameters, or all thresholds $\{r^{(i)} = (r_1^{(i)}, ..., r_s^{(i)})\}$ could be drawn in one block, in the latter case the acceptance rate is lower.

We generate

$$r_j^{(i)} \sim N(r_j^{(i-1)}, stdr_j^{(i-1)})$$

where $stdr_j^{(i-1)}$ is initially selected as a constant C_0, such that the proposal normal distribution covers all threshold parameter space (e.g, C_0=quarter-distance between upper and lower bound for each regime). After sufficient number of draws mmm we set $stdr_j^{(i-1)}$ equal to the standard deviation of the sample of accepted draws $\{r_j^{(l)}, \; l = 1, ..i - 1\}$ multiplied by a scaling constant C. The variance of the proposal density is therefore proportional to the variance matrix estimated from the simulation.

$$stdr^{(i)} = C * stdr(\{r^{(l)}\}), \qquad l = n_0, ..i - 1$$

Variance is adjusted using a scaling constant C, so that the acceptance rate is reasonable. Gelman et al. (2004) suggest optimal acceptance rate of 44% for 1 parameter and 23% for many parameters.

If $r_j^{(i)}$ does not satisfy the condition

$$r_j^{low} < r_j^{(i)} < r_j^{up}$$

where r_j^{low} and r_j^{up} are defined so that regimes below and above $r_j^{(i)}$ have at least m % of observations, then generate $r_j^{(i)}$ again until it falls within upper and lower bounds. We set $m = 5\%$.

We accept $r^{(i)} = (r_1^{(i)}, ..., r_s^{(i)})$ with probability

$$a_r = \min \left\{ \frac{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i)}, r^{(i)}|\text{data})}{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i)}, r^{(i-1)}|\text{data})}, 1 \right\}.$$

Otherwise set $r^{(i)} = r^{(i-1)}$.

Alternatively, one can construct a separate block and acceptance rate for each threshold.

(iv) **Fractional difference parameters**
We use the following procedure with efficient jump.

Each difference parameter $d_j^{(i)}$ $(1 \leq j \leq s+1)$ can be drawn either in a separate block given other difference parameters, or all parameters $\{d^{(i)} = (d_1^{(i)}, ..., d_{s+1}^{(i)})\}$

could be drawn in one block (in the latter case the acceptance rate is lower). We generate

$$d_j^{(i)} \sim N(d_j^{(i-1)}, stdd_j^{(i-1)})$$

where $stdd_j^{(i-1)}$ is initially selected as a constant C_0, such that the proposal normal distribution covers all parameter space (e.g, C_0=quarter-distance between upper and lower bounds (-0.5,1)). After sufficient number of draws mmm we set $stdd_j^{(i-1)}$ equal to the standard deviation of the sample of accepted draws $\{d_j^{(l)}, \ l = 1, ..i - 1\}$ multiplied by a scaling constant C. The variance of the proposal density is therefore proportional to the variance matrix estimated from the simulation.

$$stdd^{(i)} = C * stdd(\{d^{(l)}\}), \qquad l = n_0, ..i - 1$$

Variance is adjusted using a scaling constant C, so that acceptance rate is reasonable.

If $d_j^{(i)}$ does not satisfy the condition

$$-0.5 < d_j^{(i)} < 1$$

then generate $d_j^{(i)}$ again until it falls within upper and lower bounds.

We accept $d^{(i)} = (d_1^{(i)}, ..., d_{s+1}^{(i)})$ with probability

$$a_d = \min \left\{ \frac{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i)}, r^{(i)}, d^{(i)} | \text{data})}{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i)}, r^{(i)}, d^{(i-1)} | \text{data})}, \ 1 \right\}.$$

Otherwise set $d^{(i)} = d^{(i-1)}$.

Alternatively, one can construct a separate block and acceptance rate for each parameter d.

After d is drawn the series y_t is fractionally differenced using binomial expansion (2) with some limited number of lags n_{lags}. We change n_{lags} and make sure that the posterior pdf of d is insensitive to n_{lags}.[29]

After drawing threshold parameters r and filtering y_t with current draws of $d^{(j)}$ (where regime j is defined in (5)) the MCMC algorithm steps become identical to the steps in the ARMA model.

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[29]We found that $n_{lags} = 40$ which corresponds to 40 days was sufficient. Since in this paper we generally work with large samples sizes the problem of truncation does not arise.

As with any MCMC procedure, we make N draws of the parameters in each of the blocks, and we burn the first m draws. Out of the remaining $N - m$ draws, we keep every h-th draw. We check convergence by testing that the draws attain mean and covariance stationarity.[30].

Choice of the Number of Regimes and Orders

Estimation of a TARFIMA model involves the choice of (i) number of regimes and delay parameter and (ii) orders of ARMA(p, q) process in each regime. $ARFIMA(p, d, q)$ model choice involves orders p and q.

For each model the orders p, q, the number of regimes $s+1$ and the delay parameter δ are chosen using the significance of coefficients of lags, marginal likelihood, and a modified Bayesian information criterion (MBIC) discussed in Goldman and Tsurumi (2005). This criterion is a Bayesian analogue of Akaike information criterion given by

$$AIC = -2\Sigma_{j=1}^{n_{regimes}} ln(L_j(p_j, q_j, s, \delta)) + 2(\nu + 1)$$

where $ln(L_j())$ is a log-likelihood function for regime j and ν are degrees of freedom.

The modified Bayesian information criterion is given by:

$$MBIC = -2 \ln m(x) + 2(\nu + 1)$$

where the marginal likelihood $m(x)$ is computed by the Laplace-Metropolis estimation and evaluated at either posterior mean or mode.[31] The MBIC criterion is an in-sample information criterion showing overall fit of the models penalizing for additional parameters.

For the choice of number of regimes we find the smallest MBIC. In addition we perform sensitivity analysis where estimation of thresholds r_j is done with and without restriction $r^{low\ (j)} < r_j < r^{up\ (j)}$, where upper and lower bounds are determined by using the minimum percentage of observation for each regime. We look at the sensitivity of posterior densities of r_j to imposing the minimum $10\%, 5\%$ of observations and no restriction. Typically we find two regimes with at least 10% of observations in each regime.[32]

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[30]For example, we use Kolmogorov Smirnov and the filtered fluctuation tests (these tests are studied and compared in Goldman, Valieva and Tsurumi (2008))

[31]Alternatively one can use Chib and Jeliazkov (2001) estimator of marginal likelihood.

[32]We estimated models with two and three regimes, but using both in-sample and out-of-sample criteria all results were in favor of two regimes. Therefore, we present only results with two regimes in the paper.

To test whether the dynamics changes with regime one can simulate posterior distributions of differences in parameters of interest in upper and lower regimes. If there is considerable difference in some parameter's distributions it supports the hypothesis of non-linearity of series y_t.

Table 1: Summary statistics of $log\left(\sqrt{RV}\right)$ for DJ 30 Companies

This table presents mean, standard deviation, first order autocorrelation, skewness, kurtosis and fractional integration parameter d^e for log square root of realized volatilities (RV). GPH log-periodogram was used to estimate the fractional integration parameter d^e based on $m = N_{obs}^{4/5}$ lowest-frequency periodogram ordinates. The asymptotic standard error for d^e estimates is $\pi(24m)^{-1/2}$.

Firm	N obs	Mean	St.Dev.	AR(1)	Skewness	Kurtosis	d^e (as.std.)
GE	3024	0.459	0.329	0.703	0.655	4.074	0.460 (0.027)
MSFT	3024	0.666	0.332	0.681	0.344	3.967	0.465 (0.027)
HPQ	671	0.958	0.359	0.673	0.341	3.014	0.378 (0.065)
MMM	3024	0.363	0.331	0.67	0.273	3.492	0.451 (0.027)
AA	3024	0.554	0.336	0.649	0.495	3.452	0.448 (0.027)
MO	3023	0.516	0.356	0.631	0.725	4.928	0.425 (0.027)
AXP	3024	0.726	0.325	0.687	0.136	3.608	0.497 (0.027)
AIG	3024	0.374	0.347	0.689	0.343	3.271	0.502 (0.027)
BA	3024	0.599	0.312	0.620	0.371	4.227	0.404 (0.027)
CAT	3024	0.535	0.317	0.624	0.466	3.447	0.373 (0.027)
C	3010	0.689	0.306	0.657	0.582	4.324	0.467 (0.027)
KO	3024	0.493	0.280	0.685	0.241	3.641	0.498 (0.027)
DD	3024	0.550	0.300	0.654	0.564	3.645	0.407 (0.027)
XOM	1278	0.403	0.347	0.714	0.416	3.350	0.553 (0.042)
GM	3024	0.512	0.294	0.558	0.352	4.214	0.404 (0.027)
HD	3024	0.679	0.304	0.656	0.633	4.558	0.440 (0.027)
HON	3024	0.686	0.340	0.634	0.347	3.926	0.392 (0.027)
INTC	3024	0.826	0.350	0.738	0.423	3.236	0.434 (0.027)
IBM	3024	0.502	0.332	0.660	0.467	3.449	0.463 (0.027)
JPM	3024	0.513	0.377	0.761	0.850	3.899	0.457 (0.027)
JNJ	3024	0.452	0.303	0.644	0.182	3.672	0.468 (0.027)
MCD	3024	0.607	0.285	0.623	0.422	3.727	0.397 (0.027)
MRK	3024	0.526	0.309	0.635	0.302	3.546	0.414 (0.027)
PFE	3024	0.535	0.296	0.608	0.626	3.932	0.420 (0.027)
PG	3024	0.437	0.313	0.703	0.413	4.905	0.475 (0.027)
SBC	3024	0.608	0.305	0.663	0.511	4.203	0.414 (0.027)
UTX	3024	0.461	0.341	0.659	0.452	3.734	0.416 (0.027)
VZ	1130	0.657	0.368	0.657	0.232	2.941	0.458 (0.045)
WMT	3024	0.804	0.355	0.753	-0.362	2.847	0.524 (0.027)
DIS	3024	0.661	0.318	0.705	0.411	3.873	0.463 (0.027)

Table 2: Summary statistics of $log\left(\sqrt{BV}\right)$ for DJ 30 Companies

This table presents mean, standard deviation, first order autocorrelation, skewness, kurtosis and fractional integration parameter d^e for log square root of bipower variation (BV). GPH log-periodogram was used to estimate the fractional integration parameter d^e based on $m = N_{obs}^{4/5}$ lowest-frequency periodogram ordinates. The asymptotic standard error for d^e estimates is $\pi(24m)^{-1/2}$.

Firm	N obs	Mean	St.Dev.	AR(1)	Skewness	Kurtosis	d^e (as.std.)
GE	3024	0.379	0.361	0.696	0.518	3.784	0.432 (0.027)
MSFT	3024	0.612	0.343	0.660	0.316	3.816	0.445 (0.027)
HPQ	671	0.897	0.370	0.679	0.425	3.072	0.458 (0.065)
MMM	3024	0.257	0.364	0.650	0.130	3.567	0.427 (0.027)
AA	3024	0.427	0.387	0.630	0.195	3.079	0.412 (0.027)
MO	3023	0.421	0.373	0.600	0.539	4.006	0.413 (0.027)
AXP	3024	0.622	0.341	0.618	0.261	3.610	0.469 (0.027)
AIG	3024	0.282	0.382	0.669	0.278	3.102	0.476 (0.027)
BA	3024	0.498	0.344	0.590	0.214	3.873	0.394 (0.027)
CAT	3024	0.431	0.355	0.592	0.250	3.370	0.337 (0.027)
C	2650	0.589	0.334	0.609	0.470	3.955	0.447 (0.027)
KO	3024	0.395	0.297	0.626	0.340	3.595	0.474 (0.027)
DD	3024	0.459	0.331	0.641	0.413	3.266	0.377 (0.027)
XOM	1278	0.357	0.354	0.704	0.381	3.324	0.532 (0.042)
GM	3024	0.400	0.323	0.513	0.387	4.006	0.387 (0.027)
HD	3024	0.580	0.329	0.627	0.597	4.270	0.418 (0.027)
HON	3024	0.555	0.374	0.587	0.259	3.408	0.362 (0.027)
INTC	3024	0.773	0.364	0.716	0.392	3.182	0.411 (0.027)
IBM	3024	0.428	0.357	0.639	0.310	3.361	0.424 (0.027)
JPM	3024	0.414	0.423	0.756	0.601	3.379	0.443 (0.027)
JNJ	3024	0.356	0.312	0.569	0.272	3.745	0.429 (0.027)
MCD	3024	0.500	0.314	0.552	0.364	3.643	0.394 (0.027)
MRK	3024	0.430	0.312	0.555	0.309	3.621	0.381 (0.027)
PFE	3024	0.448	0.322	0.583	0.475	3.755	0.395 (0.027)
PG	3024	0.354	0.328	0.655	0.45	4.287	0.461 (0.027)
SBC	3024	0.495	0.340	0.620	0.412	3.881	0.361 (0.027)
UTX	3024	0.344	0.395	0.644	0.109	3.708	0.389 (0.027)
VZ	1130	0.586	0.373	0.664	0.190	2.854	0.463 (0.045)
WMT	3024	0.694	0.352	0.656	-0.239	3.066	0.462 (0.027)
DIS	3024	0.560	0.348	0.670	0.319	3.580	0.416 (0.027)

Table 3: TARFIMA: Posterior means and standard deviations

	Threshold r		% regime 1		max AR 1		max AR 2		$d^{(1)}$		$d^{(2)}$	
	RV	BV	RV	BV	RV	BV	RV	BV	RV	BV	RV	BV
GE	0.815	0.575	0.863	0.736	0.466	0.388	0.303	0.154	0.529	0.485	0.556	0.586
	(0.033)	(0.027)			(0.081)	(0.095)	(0.246)	(0.112)	(0.037)	(0.039)	(0.096)	(0.074)
MSFT	1.107	0.996	0.910	0.873	0.699	0.133	0.340	0.785	0.463	0.433	0.304	0.356
	(0.007)	(0.005)			(0.271)	(0.096)	(0.242)	(0.154)	(0.046)	(0.028)	(0.097)	(0.058)
HPQ	0.807	0.839	0.354	0.450	0.324	0.326	0.214	0.281	0.513	0.422	0.629	0.648
	(0.176)	(0.116)			(0.210)	(0.222)	(0.163)	(0.189)	(0.223)	(0.124)	(0.130)	(0.154)
MMM	0.690	0.483	0.851	0.756	0.369	0.294	0.103	0.224	0.533	0.467	0.518	0.554
	(0.058)	(0.068)			(0.097)	(0.087)	(0.075)	(0.130)	(0.054)	(0.043)	(0.122)	(0.084)
AA	0.933	0.600	0.866	0.688	0.264	0.266	0.176	0.102	0.431	0.420	0.559	0.655
	(0.062)	(0.147)			(0.097)	(0.117)	(0.141)	(0.065)	(0.055)	(0.062)	(0.070)	(0.119)
MO	0.784	0.581	0.795	0.686	0.254	0.410	0.167	0.151	0.440	0.401	0.411	0.432
	(0.031)	(0.030)			(0.157)	(0.190)	(0.122)	(0.111)	(0.039)	(0.046)	(0.062)	(0.047)
AXP	1.083	0.865	0.883	0.792	0.982	0.429	0.187	0.140	0.320	0.498	0.573	0.565
	(0.007)	(0.025)			(0.018)	(0.078)	(0.091)	(0.099)	(0.025)	(0.056)	(0.092)	(0.078)
AIG	0.463	0.623	0.635	0.809	0.219	0.265	0.197	0.164	0.461	0.515	0.635	0.619
	(0.091)	(0.169)			(0.107)	(0.079)	(0.141)	(0.108)	(0.047)	(0.055)	(0.103)	(0.091)
BA	0.965	0.897	0.894	0.891	0.519	0.338	0.512	0.353	0.361	0.427	0.476	0.516
	(0.006)	(0.013)			(0.267)	(0.152)	(0.244)	(0.221)	(0.021)	(0.038)	(0.073)	(0.092)
CAT	0.751	0.107	0.773	0.170	0.449	0.155	0.123	0.371	0.484	0.240	0.480	0.630
	(0.094)	(0.197)			(0.087)	(0.113)	(0.083)	(0.074)	(0.056)	(0.082)	(0.062)	(0.101)
C	0.898	0.949	0.786	0.869	0.243	0.248	0.267	0.331	0.463	0.451	0.417	0.576
	(0.010)	(0.040)			(0.152)	(0.102)	(0.201)	(0.240)	(0.049)	(0.038)	(0.093)	(0.097)
KO	0.790	0.733	0.876	0.886	0.291	0.379	0.273	0.246	0.517	0.502	0.576	0.567
	(0.011)	(0.028)			(0.110)	(0.082)	(0.228)	(0.192)	(0.043)	(0.045)	(0.100)	(0.098)
DD	0.830	0.749	0.834	0.821	0.435	0.448	0.126	0.078	0.483	0.508	0.547	0.564
	(0.038)	(0.012)			(0.115)	(0.062)	(0.108)	(0.062)	(0.047)	(0.038)	(0.092)	(0.083)
XOM	0.838	0.715	0.884	0.845	0.158	0.163	0.452	0.308	0.593	0.625	0.815	0.678
	(0.042)	(0.095)			(0.087)	(0.105)	(0.208)	(0.187)	(0.070)	(0.093)	(0.161)	(0.171)
GM	0.811	0.680	0.874	0.833	0.252	0.245	0.152	0.208	0.421	0.371	0.595	0.595
	(0.005)	(0.012)			(0.106)	(0.113)	(0.115)	(0.109)	(0.039)	(0.036)	(0.115)	(0.110)
HD	0.915	0.812	0.823	0.790	0.350	0.425	0.180	0.115	0.461	0.427	0.604	0.547
	(0.017)	(0.034)			(0.122)	(0.128)	(0.140)	(0.082)	(0.060)	(0.052)	(0.107)	(0.087)
HON	1.079	0.926	0.893	0.854	0.452	0.443	0.209	0.125	0.560	0.527	0.638	0.608
	(0.020)	(0.038)			(0.057)	(0.071)	(0.247)	(0.098)	(0.054)	(0.049)	(0.119)	(0.097)
INTC	1.071	1.058	0.770	0.789	0.337	0.301	0.324	0.267	0.437	0.468	0.563	0.512
	(0.012)	(0.051)			(0.158)	(0.159)	(0.157)	(0.188)	(0.037)	(0.051)	(0.098)	(0.102)
IBM	0.817	0.752	0.835	0.825	0.244	0.205	0.223	0.420	0.407	0.387	0.302	0.343
	(0.027)	(0.014)			(0.151)	(0.166)	(0.155)	(0.192)	(0.032)	(0.028)	(0.065)	(0.068)
JPM	0.995	0.958	0.878	0.883	0.394	0.407	0.384	0.316	0.500	0.537	0.549	0.642
	(0.036)	(0.010)			(0.095)	(0.074)	(0.251)	(0.170)	(0.037)	(0.034)	(0.107)	(0.103)
JNJ	0.785	0.690	0.884	0.876	0.320	0.279	0.212	0.228	0.491	0.444	0.484	0.524
	(0.015)	(0.037)			(0.112)	(0.104)	(0.146)	(0.211)	(0.045)	(0.036)	(0.076)	(0.099)
MCD	0.665	0.702	0.606	0.743	0.495	0.262	0.527	0.359	0.418	0.412	0.384	0.426
	(0.005)	(0.146)			(0.156)	(0.132)	(0.209)	(0.180)	(0.049)	(0.049)	(0.043)	(0.108)
MRK	0.718	0.753	0.736	0.853	0.468	0.437	0.334	0.254	0.437	0.473	0.468	0.567
	(0.056)	(0.045)			(0.184)	(0.116)	(0.164)	(0.180)	(0.070)	(0.063)	(0.066)	(0.102)
PFE	0.772	0.790	0.812	0.867	0.413	0.422	0.189	0.181	0.453	0.475	0.331	0.342
	(0.004)	(0.024)			(0.158)	(0.087)	(0.114)	(0.171)	(0.048)	(0.042)	(0.072)	(0.062)
PG	0.798	0.729	0.896	0.886	0.457	0.373	0.290	0.263	0.580	0.507	0.553	0.525
	(0.007)	(0.017)			(0.081)	(0.104)	(0.219)	(0.152)	(0.050)	(0.040)	(0.086)	(0.085)
SBC	0.968	0.853	0.889	0.866	0.415	0.365	0.105	0.096	0.534	0.475	0.476	0.456
	(0.015)	(0.030)			(0.079)	(0.071)	(0.085)	(0.073)	(0.064)	(0.039)	(0.089)	(0.080)
UTX	0.866	0.759	0.875	0.854	0.367	0.391	0.205	0.162	0.489	0.484	0.558	0.614
	(0.037)	(0.058)			(0.078)	(0.065)	(0.143)	(0.139)	(0.039)	(0.042)	(0.088)	(0.122)
VZ	0.977	0.936	0.814	0.837	0.280	0.287	0.398	0.382	0.643	0.625	0.654	0.678
	(0.092)	(0.081)			(0.107)	(0.115)	(0.205)	(0.236)	(0.103)	(0.120)	(0.158)	(0.136)
WMT	0.837	0.571	0.467	0.335	0.180	0.119	0.263	0.142	0.536	0.518	0.389	0.466
	(0.030)	(0.079)			(0.099)	(0.076)	(0.219)	(0.090)	(0.064)	(0.064)	(0.054)	(0.081)
DIS	0.929	0.939	0.824	0.873	0.423	0.375	0.126	0.110	0.584	0.555	0.533	0.551
	(0.078)	(0.031)			(0.113)	(0.064)	(0.125)	(0.090)	(0.057)	(0.040)	(0.080)	(0.103)

Table 4: TARFIMA: Posterior 1 day forecast MSE and ME

| | 1 day MSE | | | | 1 day ME | | | |
| | ave mode | | ave median | | ave mode | | ave median | |
	RV	BV	RV	BV	RV	BV	RV	BV
GE	0.035	0.031	0.060	0.059	0.001	-0.056	0.035	-0.017
MSFT	0.049	0.049	0.084	0.083	0.014	-0.045	0.057	0.001
HPQ	0.160	0.183	0.218	0.195	0.048	-0.274	0.120	0.016
MMM	0.047	0.053	0.075	0.080	-0.021	-0.088	0.016	-0.041
AA	0.132	0.147	0.193	0.218	-0.088	-0.191	-0.021	-0.111
MO	0.151	0.156	0.192	0.193	-0.025	-0.082	0.026	-0.022
AXP	0.055	0.054	0.080	0.077	0.012	-0.016	0.047	0.025
AIG	0.155	0.167	0.193	0.203	-0.049	-0.127	0.000	-0.071
BA	0.059	0.061	0.105	0.106	0.016	-0.087	0.068	-0.023
CAT	0.070	0.076	0.108	0.118	-0.019	-0.101	0.029	-0.034
C	0.038	0.038	0.066	0.066	0.037	-0.028	0.071	0.014
KO	0.046	0.046	0.068	0.071	0.006	-0.051	0.035	-0.017
DD	0.040	0.040	0.069	0.069	0.022	-0.050	0.056	-0.007
XOM	0.038	0.039	0.057	0.059	-0.019	-0.080	0.013	-0.047
GM	0.059	0.060	0.097	0.100	0.005	-0.070	0.050	-0.015
HD	0.058	0.059	0.094	0.096	0.019	-0.059	0.060	-0.012
HON	0.153	0.167	0.204	0.219	-0.039	-0.134	0.027	-0.057
INTC	0.069	0.074	0.130	0.131	0.003	-0.098	0.058	-0.036
IBM	0.036	0.036	0.061	0.060	0.028	-0.019	0.066	0.027
JPM	0.070	0.075	0.103	0.109	-0.008	-0.094	0.031	-0.048
JNJ	0.058	0.059	0.083	0.084	0.006	-0.043	0.040	-0.004
MCD	0.101	0.105	0.143	0.148	0.014	-0.080	0.054	-0.025
MRK	0.100	0.107	0.142	0.145	-0.010	-0.090	0.037	-0.040
PFE	0.229	0.238	0.264	0.270	-0.023	-0.107	0.020	-0.058
PG	0.023	0.024	0.042	0.044	-0.002	-0.047	0.024	-0.014
SBC	0.053	0.055	0.091	0.095	0.006	-0.066	0.046	-0.017
UTX	0.043	0.045	0.077	0.083	-0.013	-0.089	0.032	-0.030
VZ	0.059	0.062	0.102	0.102	-0.031	-0.095	0.028	-0.039
WMT	0.029	0.029	0.057	0.057	0.006	-0.067	0.048	-0.024
DIS	0.105	0.111	0.143	0.149	-0.011	-0.093	0.029	-0.042

Table 5: TARFIMA: Posterior 5 day forecast MSE, ME and in sample MBIC

| | 5 day MSE | | | | 5 day ME | | | | MBIC | |
| | ave mode | | ave median | | ave mode | | ave median | | at mean | |
	RV	BV	RV	BV	RV	BV	RV	BV	RV	BV
GE	0.075	0.075	0.095	0.092	0.064	0.005	0.080	0.024	-5457.899	-4931.195
MSFT	0.117	0.115	0.157	0.149	0.102	0.052	0.131	0.080	-5098.108	-4762.122
HPQ	0.303	0.285	0.411	0.315	0.263	-0.067	0.284	0.132	-438.873	n/a
MMM	0.093	0.100	0.115	0.118	0.034	-0.032	0.052	-0.008	-5156.464	-4559.120
AA	0.233	0.268	0.275	0.312	-0.015	-0.124	0.012	-0.075	-5070.004	-4294.414
MO	0.238	0.243	0.275	0.274	0.059	0.001	0.088	0.036	-4534.781	-4061.911
AXP	0.095	0.097	0.121	0.123	0.081	0.070	0.103	0.091	-5293.978	-4638.774
AIG	0.245	0.260	0.272	0.284	-0.003	-0.075	0.028	-0.044	-5058.408	-4484.265
BA	0.136	0.138	0.184	0.176	0.110	0.002	0.143	0.036	-5041.389	-4410.704
CAT	0.137	0.151	0.170	0.185	0.051	-0.018	0.073	0.015	-5086.066	-4357.029
C	0.088	0.088	0.119	0.116	0.120	0.055	0.138	0.078	-5419.037	-4708.990
KO	0.083	0.087	0.103	0.104	0.059	0.007	0.077	0.023	-6023.571	-5351.793
DD	0.084	0.084	0.111	0.105	0.090	0.020	0.108	0.037	-5574.055	-5032.803
XOM	0.071	0.073	0.089	0.087	0.026	-0.035	0.045	-0.017	-1757.287	-1692.234
GM	0.123	0.129	0.157	0.157	0.082	0.013	0.104	0.036	-5134.646	-4455.786
HD	0.119	0.121	0.157	0.152	0.101	0.023	0.125	0.049	-5518.176	-4920.435
HON	0.245	0.268	0.293	0.305	0.050	-0.044	0.083	-0.008	-4756.158	-4053.159
INTC	0.167	0.167	0.224	0.213	0.096	-0.004	0.131	0.031	-5306.304	-4954.672
IBM	0.079	0.079	0.107	0.105	0.107	0.064	0.130	0.090	-4951.845	-4459.860
JPM	0.125	0.132	0.153	0.155	0.058	-0.028	0.078	-0.007	-5284.036	-4692.975
JNJ	0.101	0.101	0.123	0.121	0.072	0.028	0.089	0.047	-5423.723	-4875.457
MCD	0.165	0.175	0.198	0.205	0.084	-0.004	0.105	0.025	-5603.548	-4751.838
MRK	0.176	0.178	0.212	0.205	0.067	-0.013	0.093	0.010	-5253.098	-4795.385
PFE	0.299	0.306	0.329	0.332	0.043	-0.037	0.065	-0.014	-5316.509	-4777.577
PG	0.054	0.057	0.073	0.073	0.043	0.009	0.061	0.027	-5678.184	-5108.187
SBC	0.113	0.119	0.145	0.149	0.079	0.013	0.100	0.036	-5603.506	-4853.812
UTX	0.097	0.109	0.125	0.135	0.054	-0.015	0.076	0.011	-4970.769	-4154.652
VZ	0.128	0.128	0.168	0.158	0.048	-0.019	0.079	0.009	-1260.425	-1245.345
WMT	0.075	0.074	0.104	0.094	0.077	-0.003	0.099	0.020	-5604.597	-4854.315
DIS	0.162	0.167	0.194	0.196	0.057	-0.017	0.075	0.004	-5696.724	-5010.914

Table 6: ARFIMA: Posterior means and standard deviations

	max AR root		fractional parameter d	
	RV	BV	RV	BV
GE	0.972	0.312	0.361	0.569
	(0.015)	(0.108)	(0.024)	(0.064)
MSFT	0.389	0.462	0.520	0.497
	(0.090)	(0.199)	(0.045)	(0.074)
HPQ	0.568	0.963	0.342	0.124
	(0.369)	(0.038)	(0.251)	(0.137)
MMM	0.353	0.363	0.630	0.626
	(0.055)	(0.053)	(0.071)	(0.066)
AA	0.975	0.253	0.199	0.641
	(0.008)	(0.044)	(0.046)	(0.059)
MO	0.989	0.263	0.312	0.490
	(0.007)	(0.126)	(0.021)	(0.047)
AXP	0.965	0.966	0.348	0.293
	(0.017)	(0.013)	(0.033)	(0.032)
AIG	0.238	0.235	0.586	0.616
	(0.085)	(0.053)	(0.048)	(0.060)
BA	0.968	0.411	0.331	0.535
	(0.026)	(0.085)	(0.035)	(0.062)
CAT	0.986	0.370	0.273	0.691
	(0.007)	(0.044)	(0.027)	(0.060)
C	0.939	0.957	0.335	0.281
	(0.022)	(0.017)	(0.038)	(0.031)
KO	0.956	0.343	0.328	0.573
	(0.017)	(0.067)	(0.039)	(0.052)
DD	0.988	0.363	0.308	0.653
	(0.008)	(0.048)	(0.025)	(0.056)
XOM	0.965	0.118	0.132	0.640
	(0.016)	(0.084)	(0.111)	(0.080)
GM	0.957	0.319	0.238	0.530
	(0.013)	(0.071)	(0.038)	(0.053)
HD	0.950	0.959	0.331	0.303
	(0.021)	(0.014)	(0.032)	(0.032)
HON	0.982	0.994	0.298	0.279
	(0.015)	(0.007)	(0.042)	(0.024)
INTC	0.995	0.994	0.399	0.368
	(0.004)	(0.005)	(0.019)	(0.020)
IBM	0.334	0.445	0.470	0.516
	(0.242)	(0.115)	(0.053)	(0.050)
JPM	0.347	0.321	0.649	0.688
	(0.059)	(0.053)	(0.055)	(0.062)
JNJ	0.288	0.248	0.558	0.517
	(0.092)	(0.085)	(0.052)	(0.045)
MCD	0.976	0.981	0.300	0.244
	(0.013)	(0.009)	(0.031)	(0.029)
MRK	0.981	0.385	0.305	0.567
	(0.009)	(0.059)	(0.027)	(0.063)
PFE	0.966	0.976	0.313	0.283
	(0.016)	(0.013)	(0.029)	(0.032)
PG	0.390	0.332	0.619	0.598
	(0.065)	(0.061)	(0.067)	(0.059)
SBC	0.981	0.325	0.287	0.689
	(0.008)	(0.041)	(0.028)	(0.060)
UTX	0.280	0.333	0.583	0.630
	(0.074)	(0.053)	(0.048)	(0.060)
VZ	0.973	0.237	0.109	0.692
	(0.012)	(0.085)	(0.076)	(0.098)
WMT	0.985	0.974	0.280	0.173
	(0.007)	(0.008)	(0.040)	(0.051)
DIS	0.989	0.992	0.311	0.269
	(0.005)	(0.005)	(0.026)	(0.028)

Table 7: ARFIMA: Posterior forecast MSE and ME

	1 day MSE				1 day ME			
	ave mode		ave median		ave mode		ave median	
	RV	BV	RV	BV	RV	BV		
GE	0.035	0.037	0.061	0.062	0.005	-0.059	0.033	-0.024
MSFT	0.048	0.052	0.085	0.084	-0.0002	-0.055	0.049	-0.015
HPQ	0.148	0.146	0.207	0.201	-0.007	-0.092	0.059	-0.028
MMM	0.052	0.058	0.076	0.081	-0.028	-0.092	0.006	-0.052
AA	0.135	0.148	0.193	0.210	-0.064	-0.168	-0.012	-0.099
MO	0.162	0.164	0.195	0.196	-0.050	-0.093	-0.003	-0.039
AXP	0.056	0.056	0.080	0.077	0.027	-0.024	0.057	0.013
AIG	0.166	0.178	0.201	0.216	-0.051	-0.124	-0.010	-0.075
BA	0.064	0.067	0.102	0.108	-0.012	-0.093	0.037	-0.038
CAT	0.075	0.083	0.109	0.118	-0.035	-0.104	0.006	-0.052
C	0.041	0.042	0.066	0.067	0.033	-0.039	0.065	0.000
KO	0.047	0.048	0.068	0.070	0.014	-0.058	0.038	-0.030
DD	0.042	0.044	0.067	0.070	-0.008	-0.064	0.021	-0.028
XOM	0.040	0.040	0.059	0.059	-0.003	-0.078	0.024	-0.049
GM	0.065	0.070	0.098	0.103	-0.001	-0.091	0.039	-0.040
HD	0.064	0.067	0.096	0.099	0.012	-0.074	0.049	-0.028
HON	0.161	0.173	0.204	0.220	-0.035	-0.140	0.018	-0.070
INTC	0.076	0.083	0.129	0.136	-0.036	-0.126	0.015	-0.072
IBM	0.039	0.039	0.061	0.061	0.017	-0.037	0.053	0.003
JPM	0.073	0.080	0.102	0.109	-0.023	-0.103	0.013	-0.061
JNJ	0.060	0.061	0.083	0.084	-0.003	-0.054	0.026	-0.021
MCD	0.109	0.111	0.145	0.150	-0.010	-0.095	0.027	-0.050
MRK	0.103	0.112	0.141	0.146	-0.028	-0.101	0.009	-0.060
PFE	0.229	0.240	0.264	0.270	-0.028	-0.112	0.011	-0.068
PG	0.025	0.027	0.042	0.045	-0.008	-0.057	0.017	-0.028
SBC	0.057	0.062	0.091	0.097	-0.005	-0.088	0.029	-0.044
UTX	0.046	0.050	0.077	0.084	-0.019	-0.090	0.020	-0.038
VZ	0.066	0.067	0.103	0.103	-0.018	-0.097	0.031	-0.050
WMT	0.031	0.032	0.055	0.057	0.003	-0.046	0.033	-0.009
DIS	0.104	0.114	0.140	0.150	-0.024	-0.107	0.010	-0.063

Table 8: ARFIMA: Posterior 5 day forecast MSE, ME and in sample MBIC

| | 5 day MSE | | | | 5 day ME | | | | MBIC | |
| | ave mode | | ave median | | ave mode | | ave median | | at mean | |
	RV	BV	RV	BV	RV	BV	RV	BV	RV	BV
GE	0.076	0.079	0.100	0.100	0.053	0.001	0.079	0.023	-5382.513	-4900.785
MSFT	0.118	0.116	0.157	0.148	0.093	0.024	0.119	0.054	-5086.452	-4772.736
HPQ	0.264	0.253	0.336	0.310	0.108	0.012	0.152	0.056	-240.275	-451.207
MMM	0.095	0.101	0.117	0.121	0.024	-0.037	0.041	-0.017	-5169.279	-4571.991
AA	0.238	0.265	0.284	0.310	0.001	-0.085	0.028	-0.055	-5088.268	-4287.819
MO	0.241	0.243	0.273	0.276	0.020	-0.006	0.043	0.022	-4523.473	-4082.801
AXP	0.100	0.098	0.128	0.123	0.105	0.057	0.123	0.080	-5300.006	-4646.576
AIG	0.252	0.267	0.280	0.293	0.004	-0.063	0.026	-0.038	-5071.024	-4494.319
BA	0.134	0.141	0.174	0.178	0.065	-0.011	0.094	0.018	-5041.429	-4425.546
CAT	0.140	0.152	0.170	0.183	0.022	-0.035	0.043	-0.012	-5108.996	-4406.671
C	0.092	0.091	0.126	0.119	0.120	0.044	0.139	0.068	-5381.378	-4680.505
KO	0.085	0.086	0.106	0.103	0.073	-0.004	0.087	0.011	-6060.680	-5398.072
DD	0.081	0.086	0.104	0.107	0.044	-0.004	0.059	0.012	-5564.820	-5034.491
XOM	0.076	0.075	0.097	0.089	0.049	-0.032	0.068	-0.017	-1770.894	-1718.093
GM	0.126	0.132	0.161	0.161	0.074	-0.014	0.093	0.008	-5118.971	-4446.628
HD	0.124	0.127	0.163	0.160	0.090	0.004	0.114	0.032	-5474.880	-4888.519
HON	0.249	0.271	0.294	0.311	0.039	-0.050	0.072	-0.017	-4762.183	-4056.860
INTC	0.167	0.174	0.216	0.215	0.027	-0.053	0.062	-0.023	-5318.941	-4970.278
IBM	0.081	0.079	0.109	0.103	0.096	0.038	0.121	0.063	-4937.264	-4457.169
JPM	0.125	0.133	0.155	0.159	0.037	-0.041	0.057	-0.019	-5241.085	-4672.612
JNJ	0.101	0.104	0.121	0.121	0.058	0.012	0.072	0.028	-5442.732	-4890.312
MCD	0.167	0.175	0.199	0.205	0.056	-0.025	0.076	-0.002	-5600.779	-4789.930
MRK	0.172	0.178	0.204	0.204	0.035	-0.031	0.056	-0.012	-5266.431	-4807.444
PFE	0.303	0.311	0.336	0.337	0.038	-0.050	0.061	-0.023	-5287.426	-4774.597
PG	0.057	0.059	0.075	0.074	0.040	-0.006	0.054	0.009	-5618.292	-5097.404
SBC	0.114	0.122	0.145	0.150	0.058	-0.017	0.076	0.000	-5580.262	-4872.616
UTX	0.099	0.110	0.127	0.137	0.044	-0.019	0.064	0.005	-4964.188	-4156.829
VZ	0.134	0.129	0.171	0.157	0.058	-0.030	0.084	-0.006	-1289.221	-1270.668
WMT	0.071	0.076	0.094	0.100	0.061	0.025	0.077	0.046	-5599.403	-4884.271
DIS	0.160	0.169	0.192	0.199	0.035	-0.039	0.054	-0.019	-5667.714	-4998.018

Table 9: ARMA(1,1): Posterior means, standard deviation and forecasts for RV

Firm	Nobs	max AR root	1 day MSE		1 day ME		5 day MSE		5 day ME		MBIC at mean
			mode	median	mode	median	mode	median	mode	median	
GE	3024	0.954 (0.008)	0.037	0.062	0.019	0.047	0.082	0.108	0.075	0.104	-5432.68
MSFT	3024	0.937 (0.010)	0.053	0.090	0.052	0.087	0.133	0.183	0.144	0.182	-5136.42
HPQ	671	0.953 (0.028)	0.161	0.212	0.012	0.081	0.285	0.359	0.135	0.189	-526.10
MMM	3024	0.965 (0.007)	0.053	0.076	-0.007	0.022	0.097	0.121	0.036	0.061	-5196.56
AA	3024	0.974 (0.006)	0.135	0.193	-0.054	-0.007	0.239	0.284	-0.005	0.029	-5132.10
MO	3023	0.953 (0.009)	0.162	0.201	-0.009	0.035	0.249	0.292	0.064	0.102	-4556.25
AXP	3024	0.952 (0.009)	0.060	0.084	0.061	0.087	0.114	0.148	0.150	0.175	-5345.12
AIG	3024	0.962 (0.007)	0.175	0.205	-0.034	0.002	0.254	0.284	0.008	0.039	-5124.33
BA	3024	0.943 (0.010)	0.064	0.107	0.023	0.064	0.142	0.188	0.097	0.133	-5104.71
CAT	3024	0.975 (0.006)	0.079	0.110	-0.014	0.022	0.143	0.175	0.036	0.062	-5146.20
C	3010	0.942 (0.010)	0.046	0.072	0.067	0.096	0.108	0.149	0.164	0.192	-5426.42
KO	3024	0.954 (0.008)	0.049	0.070	0.034	0.054	0.089	0.113	0.090	0.110	-6146.30
DD	3024	0.966 (0.007)	0.045	0.069	0.024	0.051	0.087	0.114	0.080	0.102	-5609.12
XOM	1278	0.971 (0.011)	0.042	0.060	-0.001	0.027	0.077	0.098	0.044	0.068	-1840.51
GM	3024	0.953 (0.009)	0.067	0.103	0.021	0.056	0.133	0.169	0.086	0.114	-5176.85
HD	3024	0.946 (0.009)	0.069	0.101	0.040	0.074	0.134	0.178	0.120	0.152	-5478.15
HON	3024	0.968 (0.007)	0.157	0.201	-0.012	0.037	0.249	0.298	0.057	0.095	-4818.35
INTC	3024	0.956 (0.008)	0.085	0.136	0.015	0.058	0.183	0.241	0.089	0.134	-5332.29
IBM	3024	0.922 (0.012)	0.043	0.066	0.058	0.090	0.097	0.134	0.153	0.187	-4950.95
JPM	3024	0.973 (0.006)	0.074	0.103	-0.010	0.021	0.129	0.159	0.036	0.065	-5281.74
JNJ	3024	0.959 (0.008)	0.063	0.086	0.023	0.047	0.106	0.129	0.080	0.102	-5504.79
MCD	3024	0.956 (0.008)	0.108	0.146	0.018	0.049	0.173	0.210	0.081	0.107	-5671.51
MRK	3024	0.960 (0.008)	0.105	0.142	0.002	0.035	0.177	0.211	0.064	0.091	-5305.99
PFE	3024	0.946 (0.010)	0.234	0.271	0.003	0.037	0.314	0.352	0.072	0.101	-5315.52
PG	3024	0.956 (0.008)	0.027	0.046	0.027	0.049	0.064	0.086	0.083	0.105	-5674.97
SBC	3024	0.971 (0.006)	0.060	0.095	0.012	0.043	0.119	0.151	0.066	0.091	-5615.86
UTX	3024	0.968 (0.006)	0.047	0.079	0.000	0.033	0.102	0.132	0.051	0.079	-5032.65
VZ	1130	0.976 (0.011)	0.068	0.106	0.000	0.047	0.141	0.182	0.073	0.106	-1316.96
WMT	3024	0.979 (0.005)	0.032	0.056	0.016	0.041	0.073	0.098	0.065	0.088	-5664.12
DIS	3024	0.972 (0.006)	0.106	0.143	0.002	0.033	0.168	0.202	0.060	0.086	-5725.56

Table 10: Mean squared errors and MBIC for RV

Firm	N obs	1 day MSE modes M1	M2	1 day MSE median M1	M2	5 day MSE modes M1	M2	5 day MSE median M1	M2	MBIC at mean M1	M2
GE	3024	0.854	0.996	0.954	0.990	0.888	0.957	0.881	0.942	1.006	0.992
MSFT	3024	0.928	0.952	0.933	0.939	0.861	0.898	0.846	0.875	0.993	0.989
HPQ	671	0.997	0.920	1.029	0.974	1.065	0.927	1.145	0.936	0.834	0.457
MMM	3024	0.902	1.000	0.984	0.994	0.958	0.974	0.951	0.965	0.992	0.995
AA	3024	0.977	0.999	1.001	0.998	0.977	0.995	0.970	1.000	0.988	0.991
MO	3023	0.930	0.999	0.957	0.967	0.956	0.969	0.941	0.936	0.995	0.993
AXP	3024	0.911	0.930	0.945	0.946	0.832	0.878	0.818	0.863	0.990	0.992
AIG	3024	0.885	0.946	0.940	0.979	0.962	0.989	0.955	0.987	0.987	0.990
BA	3024	0.927	0.999	0.989	0.959	0.957	0.942	0.978	0.925	0.988	0.988
CAT	3024	0.886	0.942	0.976	0.984	0.962	0.980	0.973	0.975	0.988	0.993
C	3010	0.823	0.899	0.911	0.918	0.817	0.857	0.802	0.850	0.999	0.992
KO	3010	0.934	0.967	0.978	0.978	0.931	0.954	0.913	0.940	0.980	0.986
DD	3024	0.892	0.928	0.988	0.971	0.960	0.929	0.975	0.915	0.994	0.992
XOM	1278	0.905	0.966	0.957	0.980	0.925	0.990	0.915	0.993	0.955	0.962
GM	3024	0.881	0.970	0.940	0.955	0.922	0.946	0.928	0.951	0.992	0.989
HD	3024	0.843	0.935	0.930	0.951	0.890	0.928	0.881	0.916	1.007	0.999
HON	3024	0.972	1.022	1.012	1.016	0.984	1.000	0.985	0.989	0.987	0.988
INTC	3024	0.816	0.893	0.955	0.947	0.914	0.913	0.928	0.894	0.995	0.997
IBM	3024	0.838	0.909	0.922	0.929	0.814	0.836	0.803	0.813	1.000	0.997
JPM	3024	0.944	0.995	0.999	0.988	0.971	0.970	0.960	0.976	1.000	0.992
JNJ	3024	0.926	0.961	0.970	0.967	0.951	0.953	0.949	0.939	0.985	0.989
MCD	3024	0.932	1.011	0.984	0.998	0.956	0.964	0.941	0.948	0.988	0.988
MRK	3024	0.944	0.981	0.997	0.990	0.995	0.973	1.002	0.964	0.990	0.993
PFE	3024	0.979	0.980	0.975	0.976	0.951	0.964	0.934	0.955	1.000	0.995
PG	3024	0.875	0.941	0.913	0.927	0.852	0.892	0.846	0.871	1.001	0.990
SBC	3024	0.890	0.956	0.960	0.964	0.946	0.962	0.963	0.964	0.998	0.994
UTX	3024	0.909	0.980	0.977	0.983	0.959	0.975	0.953	0.967	0.988	0.986
VZ	1130	0.870	0.977	0.962	0.974	0.908	0.945	0.921	0.938	0.957	0.979
WMT	3024	0.925	0.971	1.034	0.981	1.027	0.961	1.062	0.966	0.989	0.989
DIS	3024	0.989	0.976	1.001	0.980	0.969	0.954	0.963	0.951	0.995	0.990

Notes: M1= TARFIMA, M2=ARFIMA

Averages of MSE modes and medians are shown for 1 and 5 day forecasts.

MSE is relative to M3=ARMA model for which MSE is set to 1

MBIC is relative to M3=ARMA model for which it is set to 1 (here we maximize MBIC)

Table 11: Summary comparison for all models

Model	1 day MSE mode	median	5 day MSE mode	median	MBIC mean
M1	28	18	21	21	6
M2	2	10	9	9	
M3		1			24
M1=M2		1			
GE	M1	M1	M1	M1	M1
MSFT	M1	M1	M1	M1	M3
HPQ	M2	M2	M2	M2	M3

Notes: M1= TARFIMA, M2=ARFIMA, M3=ARMA

Each entry shows number of cases among 30 firms

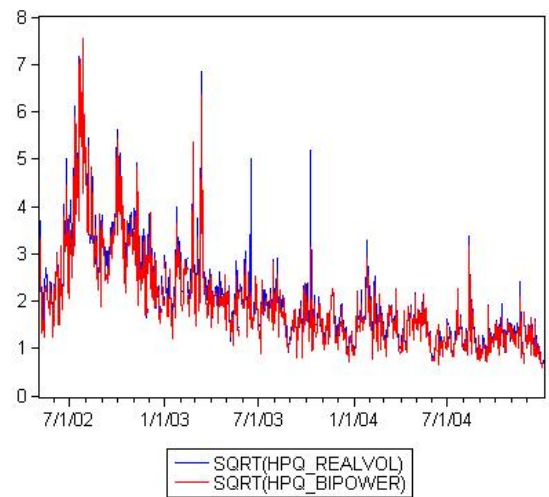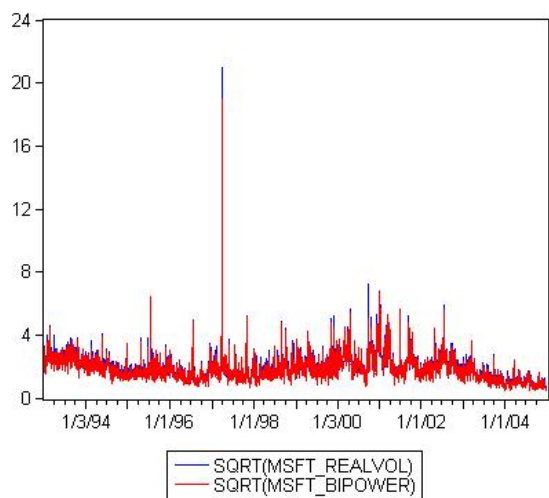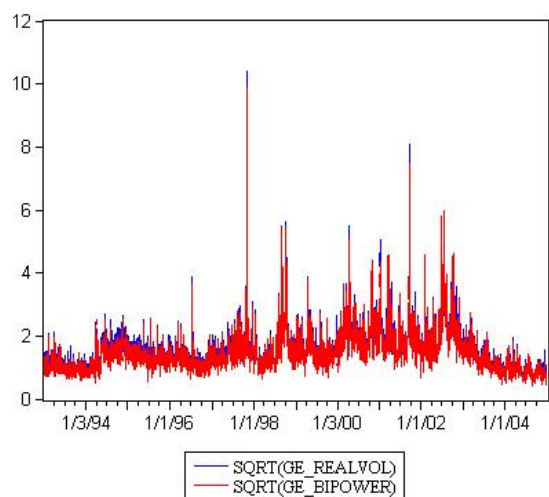when a particular model (M1, M2 or M3) was selected

Figure 1: Square Root of Realized Daily Variance and Bipower Variation for GE, MSFT, and HPQ
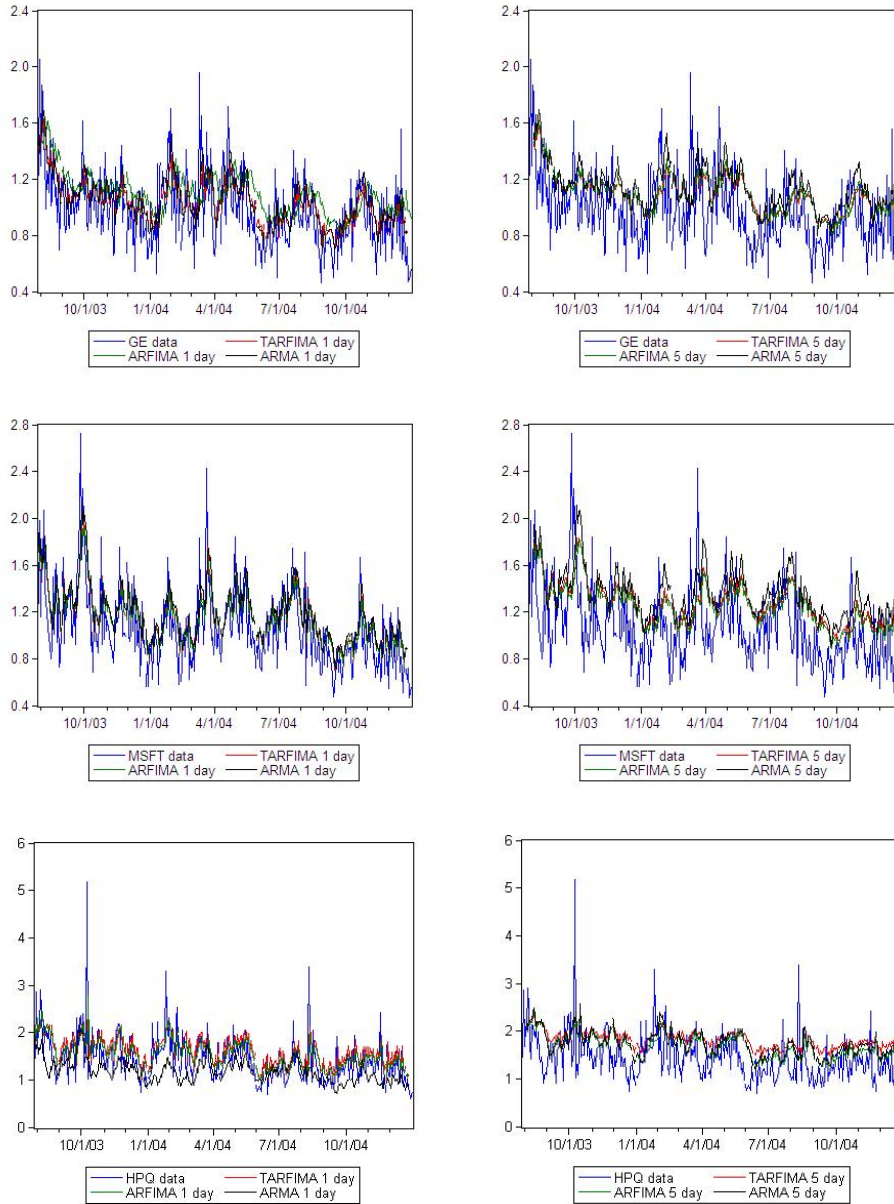
Figure 2: Posterior means of 1 and 5 day forecasts of realized volatility for GE, MSFT, and HPQ
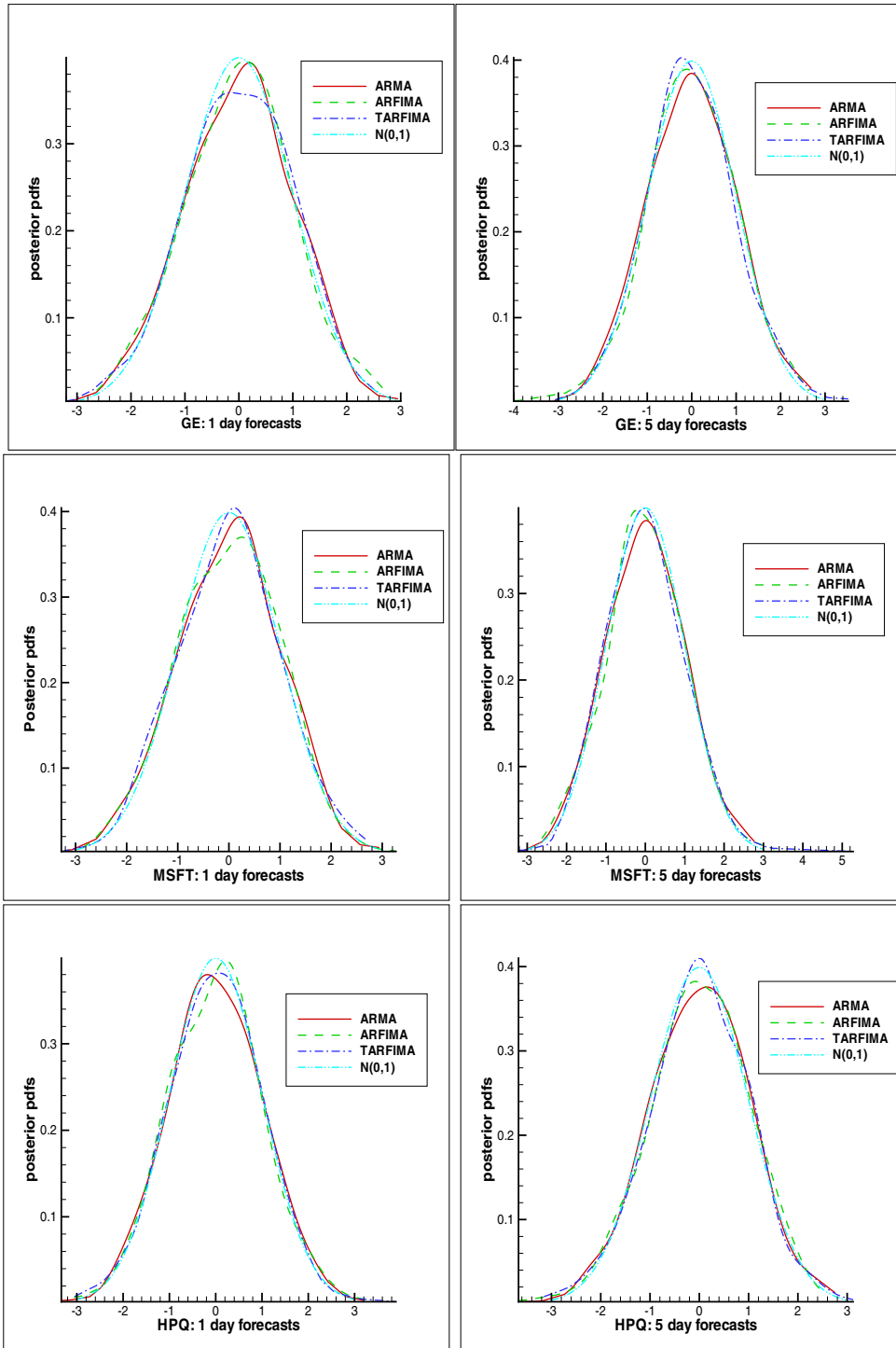
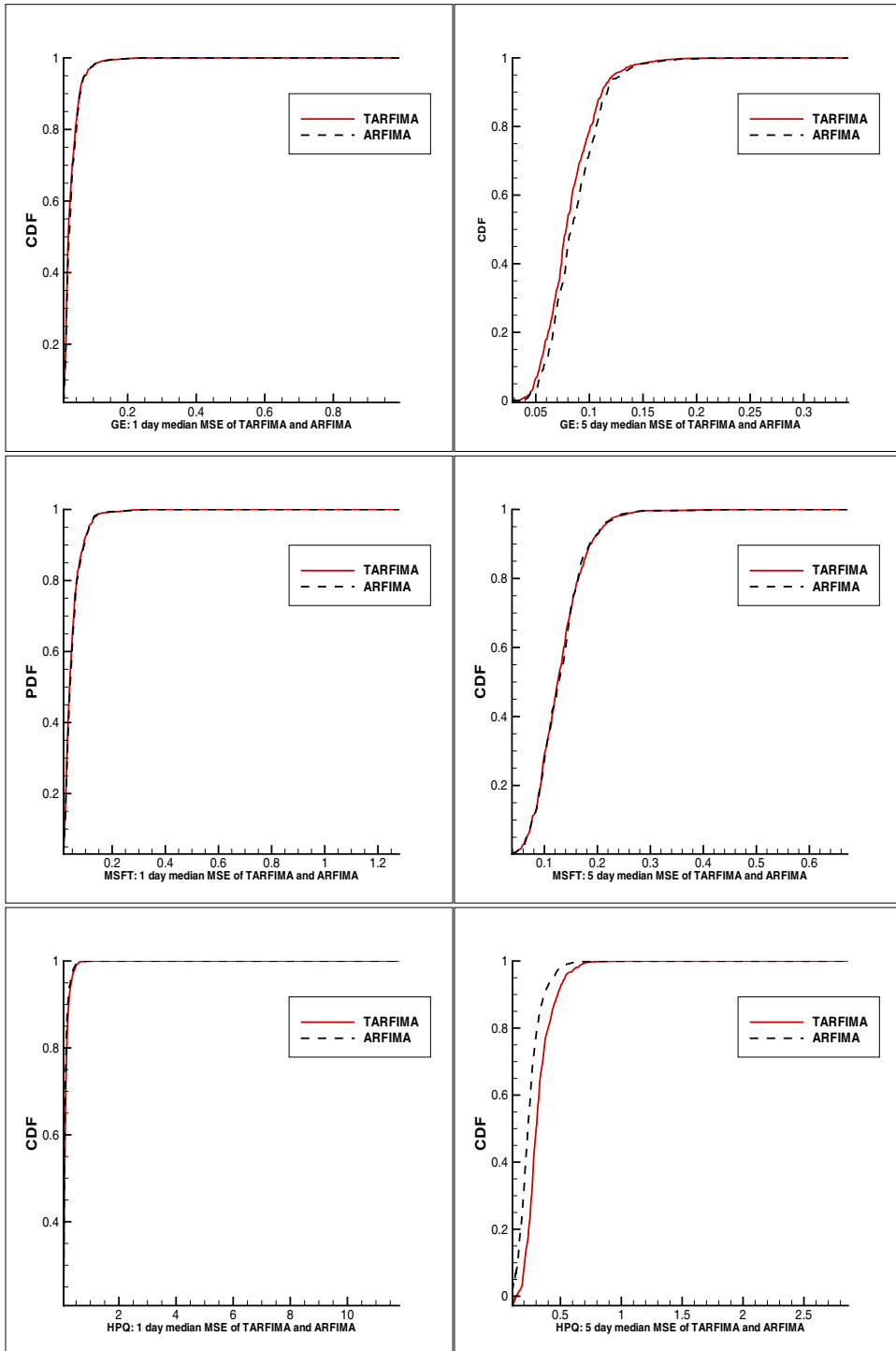Figure 3: Posterior pdfs of normalized log realized volatility forecasts for GE, MSFT and HPQ

Figure 4: Posterior cdfs of the median MSE(ARFIMA) and median MSE(TARFIMA)
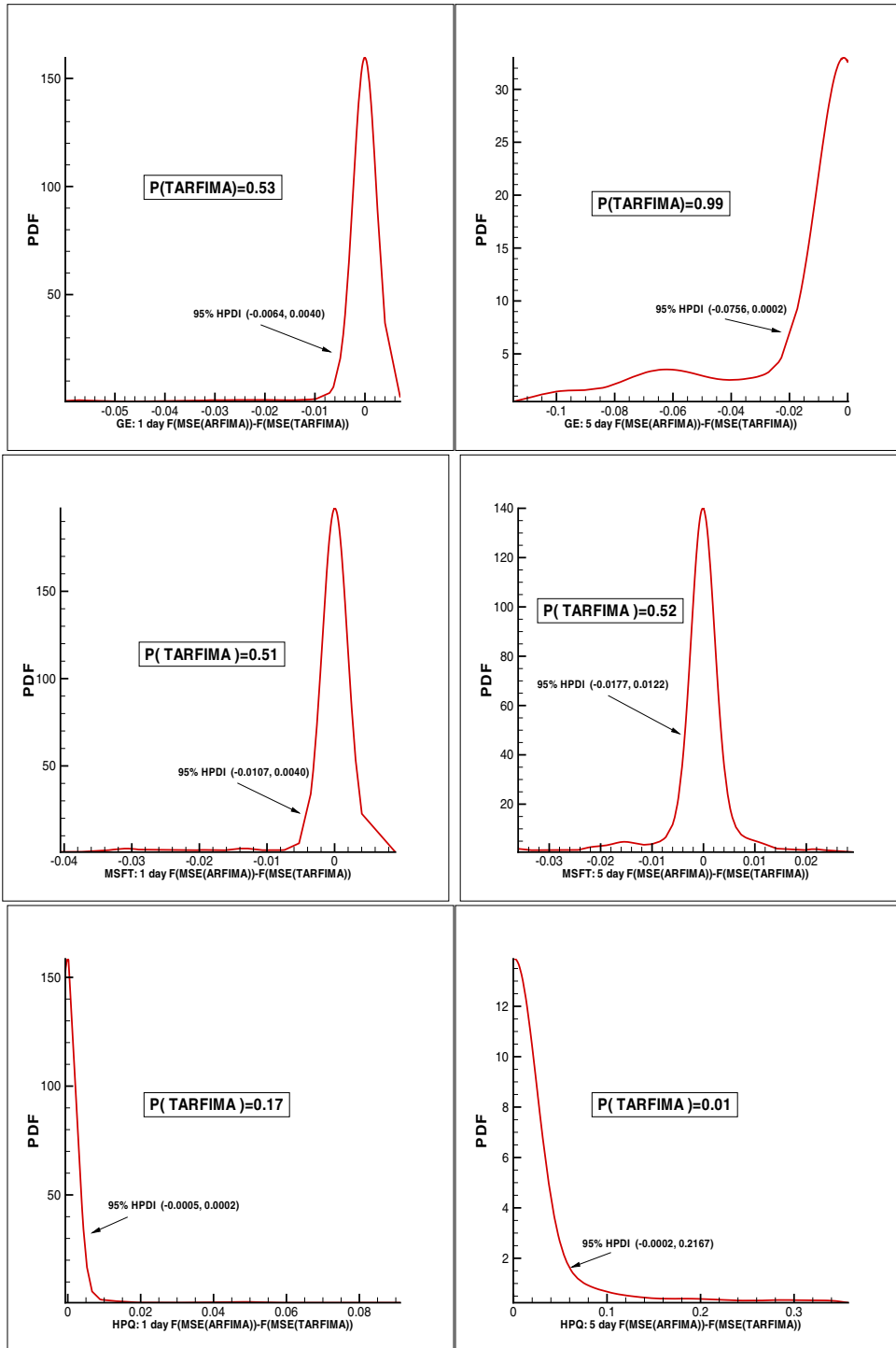
Figure 5: Posterior pdfs of the median F(MSE(ARFIMA))-F(MSE(TARFIMA))